BRITISH COLUMBIA SECONDARY SCHOOL MATHEMATICS CONTEST, 2026

Senior Final, Part B Problems & Solutions

1. A certain lottery involves a game with four identical boxes containing balls. One box has 5 black balls and 5 white balls, one has 4 black and 2 white, one has 3 black and no white, and one has 2 black and 4 white. A player first chooses a box at random, then chooses a ball within that box at random. If the ball is black, the player wins; if it is white, the player loses. Occasionally, at the start of the game, a lucky player is given the ability to redistribute the balls among the boxes before playing the game. How should the balls be distributed so that the probability of winning is as large as possible? Explain.

Solution

The probability of winning the game with the initial distribution equals

$$P = \frac{1}{4}\left(\frac{5}{5+5} + \frac{4}{4+2} + \frac{3}{3+0} + \frac{2}{2+4}\right) = \frac{5}{8}.$$

So, there are 14 black and 11 white balls in total.

The lucky player must redistribute the balls by putting b_i black balls and w_i white balls in Box i for $i = 1, 2, 3, 4$ to maximize

$$\frac{1}{4}\left(\frac{b_1}{b_1+w_1} + \frac{b_2}{b_2+w_2} + \frac{b_3}{b_3+w_3} + \frac{b_4}{b_4+w_4}\right) \tag{$*$}$$

• First we claim that in an optimal distribution, at most one box contains white balls.

To check this, assume, without loss of generality, that Box 1 contains $w_1 > 0$ white balls, Box 2 contains $w_2 > 0$ white balls and let $b_1 \le b_2$. The contribution to the sum is

$$\frac{b_1}{b_1+w_1} + \frac{b_2}{b_2+w_2}.$$

Move all the white balls from Box 1 to Box 2. The new sum becomes

$$1 + \frac{b_2}{b_2+w_1+w_2}.$$

The difference between the new sum and the old sum is

$$\left(1 - \frac{b_1}{b_1+w_1}\right) - \left(\frac{b_2}{b_2+w_2} - \frac{b_2}{b_2+w_1+w_2}\right)$$

$$= \frac{w_1}{b_1+w_1} - \frac{b_2 w_1}{(b_2+w_2)(b_2+w_1+w_2)} > 0, \text{ positive}$$

because

$$\frac{w_1}{b_1+w_1} > \frac{b_2 w_1}{(b_2+w_2)(b_1+w_1)} > \frac{b_2 w_1}{(b_2+w_2)(b_2+w_1+w_2)}.$$

So, eventually, all the white balls will be moved to only one box.

• Second, since all the white balls are in one box and the rest boxes contain only black balls, the sum $(*)$ will be maximized when the largest number of black balls are placed in the box with the white balls. It means that each box with no white balls must contain exactly one black ball.

The required distribution is

$$(b, w): \ (1,0),\ (1,0),\ (1,0),\ (11,11).$$

The probability of winning in this case equals

$$\frac{1}{4}\left(1+1+1+\frac{11}{22}\right) = \frac{7}{8}.$$

2. You have 100 large jugs, each with a capacity of 10 Litres. For each integer $n = 1, \ldots, 100$, there is a jug that contains exactly n mL of water. You are allowed to redistribute the water among the jugs by pouring water from one jug into another as many times as you like subject to the rule that, at each step in the process, the amount of water poured into a jug must equal the amount that is in the jug already (so that the effect of the added water is to double the amount of water in the jug).

   a) Is it possible to redistribute the water so that, for each $n = 1, \ldots, 100$, there is a jug that contains exactly n mL of water, but no jug contains the same amount of water as it did initially? Explain.

   b) Is it possible to redistribute the water so that five of the jugs each contain exactly 3 mL of water and, for each $n = 6, 7, \ldots, 100$, there is a jug that contains exactly n mL of water? Explain.

Solution

a) Yes. Proceed as follows

$$100 \longrightarrow 1, \quad 99 \longrightarrow 2, \quad 98 \longrightarrow 3, \quad \ldots, \quad 51 \longrightarrow 50$$

by pouring from jug A to jug B the amount that jug B initially has. The result is

$$2,\ 4,\ 6,\ \ldots,\ 100,\ 1,\ 3,\ 5,\ \ldots\ ,\ 97,\ 99.$$

b) No. Following the given rule, the quantity of jugs containing odd numbers of millilitres of water should never increase.

3. Find the value of the number p that minimizes the sum of the squares of the roots of the following equation:

$$x^2 + (p-2)x + 2(p+1) = 0.$$

Solution

Recall that in the quadratic equation $ax^2 + bx + c = 0$ the sum of the squares of the roots equals

$$\left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)^2 + \left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)^2 = \frac{b^2 - 2ac}{a^2}.$$

Therefore, in our equation the sum of the roots equals

$$\frac{(p-2)^2 - 2 \cdot 1 \cdot 2(p+1)}{1} = p^2 - 8p$$

whenever the discriminant $D = b^2 - 4ac \geq 0$. So, we require

$$D = (p-2)^2 - 8(p+1) = p^2 - 12p - 4 \geq 0.$$

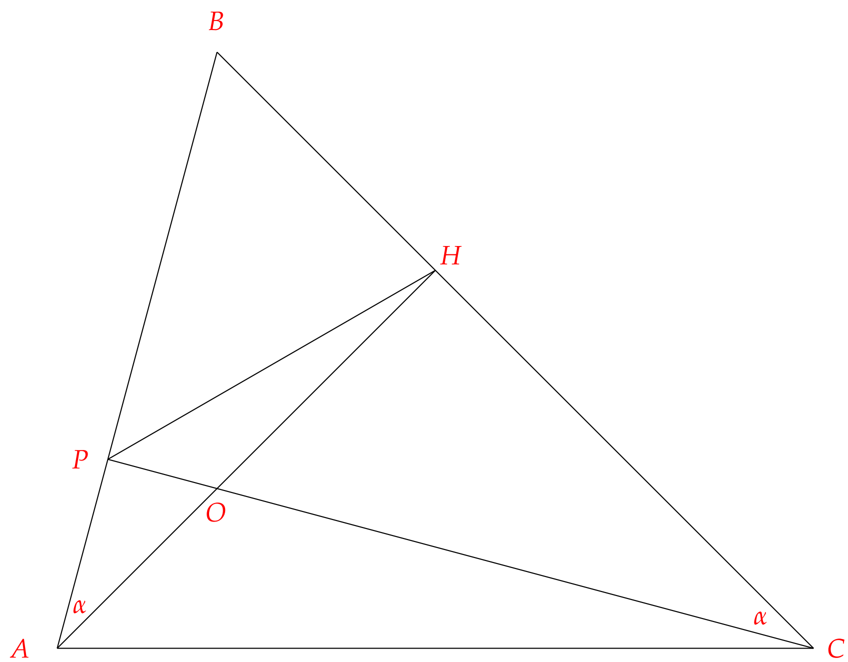
We obtain the constraints

$$(p-6)^2 - 40 \geq 0 \quad \Longrightarrow \quad p \geq 6 + 2\sqrt{10} \ \text{ or } \ p \leq 6 - 2\sqrt{10}.$$

The minimum of the parabola $p^2 - 8p$ under the given constraints will be attained in one of the limiting points. Since $6 - 2\sqrt{10}$ is closer to 4 (the x-coordinate of the vertex) than $6 + 2\sqrt{10}$ is, the answer is $p = 6 - 2\sqrt{10}$.

4. AH and CP are heights of the acute triangle ABC. What is the value of the angle B if it is known that $AC = 2HP$?

Solution



The right triangles ΔAPO and ΔCHO are similar since they contain vertical angles at the vertex O. So, $\angle PAO = \angle HCO \equiv \alpha$. From the right triangle ΔAHB, $\sin\alpha = \frac{BH}{BA}$ and from the right triangle ΔCPB, $\sin\alpha = \frac{BP}{BC}$. Since $\frac{BH}{BA} = \frac{BP}{BC}$ and the angle $\angle B$ being common, the triangles ΔBHP and ΔBAC are similar. Moreover, the scale factor equals 2 because $AC = 2HP$ by the conditions. This implies $\sin\alpha = \frac{BH}{BA} = \frac{1}{2}$ which means that $\alpha = 30^\circ$ and so, $\angle B = 60^\circ$.

5. Let us call a positive integer "nice" if it can be expressed as the sum of a three-digit number abc and a three-digit number cba, where the symbols a, b and c represent digits and the digits a and c are not zero. How many positive integers are nice?

Solution

For digits a, b and c with $a \neq 0$ and $c \neq 0$ consider the column addition of the numbers abc and cba.

$$\begin{array}{cccc} & a & b & c \\ + & c & b & a \\ \hline & \square & \square & \square \end{array} \quad \text{or} \quad \begin{array}{cccc} & a & b & c \\ + & c & b & a \\ \hline \square & \square & \square & \square \end{array}$$

We will count the number of different three-digit and four-digit nice numbers in the following cases.

Case 1. $2 \leq a + c \leq 9$ and $0 \leq b \leq 4$.

Then the sum is three-digit. Below are the possibilities for the digits:

$a+c$	$2b$	$c+a$
2, 3, 4, 5, 6, 7, 8, 9	0, 2, 4, 6 , 8	2, 3, 4, 5, 6, 7, 8, 9

So, $8 \times 5 = 40$ different numbers.

Case 2. $2 \leq a + c \leq 9$ and $5 \leq b \leq 9$. Then $2b \geq 10$ and we carry 1 over.

Then the sum is three or four-digit. Below are the possibilities for the digits:

When $a + c < 9$ the nice number is still three-digit.

$a+c+1$	$2b-10$	$c+a$
3, 4, 5, 6, 7, 8, 9	0, 2, 4, 6 , 8	2, 3, 4, 5, 6, 7, 8

When $a + c = 9$, the nice number is four-digit with 1 for the thousands and 0 for the hundredths.

	$a+c+1-10$	$2b-10$	$c+a$
1	0	0, 2, 4, 6 , 8	9

So, $7 \times 5 + 5 = 40$ different numbers.

Case 3. $10 \leq a + c \leq 18$ and $0 \leq b \leq 4$.

Then the nice number is four-digit. Below are the possibilities for the digits:

	$a+c-10$	$2b+1$	$c+a-10$
1	0, 1, 2, 3, 4, 5, 6, 7, 8	1, 3, 5, 7 , 9	0, 1, 2, 3, 4, 5, 6, 7, 8

So, $9 \times 5 = 45$ different numbers.

Case 4. $10 \leq a + c \leq 18$ and $5 \leq b \leq 9$.

Then the nice number is four-digit. Below are the possibilities for the digits:

	$a+c+1-10$	$2b+1-10$	$c+a-10$
1	1, 2, 3, 4, 5, 6, 7, 8, 9	1, 3, 5, 7 , 9	0, 1, 2, 3, 4, 5, 6, 7, 8

So, $9 \times 5 = 45$ different numbers.

In total we obtain $40 + 40 + 45 + 45 = 170$ different nice numbers.