BRITISH COLUMBIA SECONDARY SCHOOL MATHEMATICS CONTEST, 2025

Junior Final, Part B Problems & Solutions

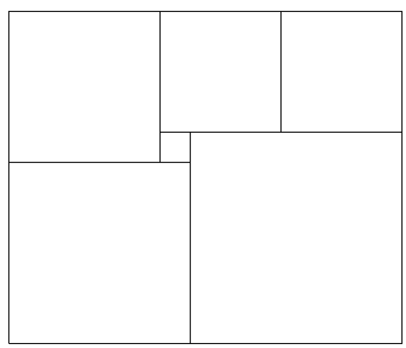
1. For some integers x and y, the number $3x + 2y$ is divisible by 23. Prove that the number $17x + 19y$ is also divisible by 23.

   Solution

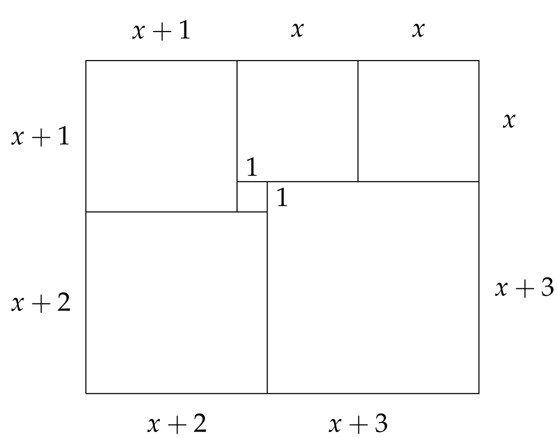
   Note that
   $$17x + 19y + 2(3x + 2y) = 23x + 23y = 23(x + y),$$
   so, divisible by 23.

   Since $3x + 2y$ is divisible by 23, the difference
   $$23(x + y) - 2(3x + 2y)$$
   which is $17x + 19y$ will be also divisible by 23.

2. The diagram shown is a rectangle that has been divided into 6 squares. If the smallest square has sides of length 1, what are the lengths of the sides of the rectangle?



   Solution



   Then $x + 1 + x + x = x + 2 + x + 3$ and so $x = 4$.

   The length of the sides of the rectangle are 13 and 11.

3. A highly composite number is a positive integer that has more divisors than any smaller positive integer. For example, 6 is a highly composite number because it has more divisors than 1,2,3,4 or 5. How many highly composite numbers are there between 10 and 50?

Solution

Note that if the prime factorization of a positive integer n is given by $n = p_1^{a_1} \cdot p_2^{a_2} \cdots p_k^{a_k}$, then n has exactly $(a_1+1)(a_2+1)\cdots(a_k+1)$ positive divisors. We may hence use the prime factorization of any integer to determine how many divisors it has. The following tables display the prime factorizations, and number of positive divisors of the first 60 positive integers:

n	fact	# div
1	N/A	1
2	prime	2
3	prime	2
4	2^2	3
5	prime	2
6	$2 \cdot 3$	4
7	prime	2
8	2^3	4
9	3^2	3
10	$2 \cdot 5$	4
11	prime	2
12	$2^2 \cdot 3$	6
13	prime	2
14	$2 \cdot 7$	4
15	$3 \cdot 5$	4
16	$2 \cdot 4$	5
17	prime	2
18	$2 \cdot 3^2$	6
19	prime	2
20	$2^2 \cdot 5$	6
21	$3 \cdot 7$	4
22	$2 \cdot 11$	4
23	prime	2
24	$2^3 \cdot 3$	8
25	5^2	3
26	$2 \cdot 13$	4
27	3^3	4
28	$2^2 \cdot 7$	6
29	prime	2
30	$2 \cdot 3 \cdot 5$	8

n	fact	# div
31	prime	2
32	2^5	6
33	$3 \cdot 11$	4
34	$2 \cdot 17$	4
35	$3 \cdot 5$	4
36	$2^2 \cdot 3^2$	9
37	prime	2
38	$2 \cdot 19$	4
39	$3 \cdot 13$	4
40	$2^3 \cdot 5$	8
41	prime	2
42	$2 \cdot 3 \cdot 7$	8
43	prime	2
44	$2^2 \cdot 11$	6
45	$3^2 \cdot 5$	6
46	$2 \cdot 23$	4
47	prime	2
48	$2^4 \cdot 3$	10
49	7^2	3
50	$2 \cdot 5^2$	6
51	$3 \cdot 17$	4
52	$2^2 \cdot 13$	6
53	prime	2
54	$2 \cdot 3^3$	8
55	$5 \cdot 11$	4
56	$2^3 \cdot 7$	8
57	$3 \cdot 19$	4
58	$2 \cdot 29$	4
59	prime	2
60	$2^2 \cdot 3 \cdot 5$	12

From this table it may be seen that $1, 2, 4, 6, 12, 24, 36, 48, 60$ are highly composite. Hence the number of highly composite numbers between 10 and 50 is **four** (12, 24, 36 and 48).

4. The number 321321321321 is written on a blackboard. What digits should be erased to obtain the largest number divisible by 9?

Solution

For each integer divisible by 9, the sum of there digits is divisible by 9. The sum of the digits of the number $a = 321321321321$ equals 24. The largest integer that is no larger than 24 and is divisible by 9 is 18. Since $24 - 18 = 6$, we must erase digits whose sum is 6. The largest number will be obtained when the least number of digits is erased. Therefore, the answer is

$$321321\cancel{3}21\cancel{3}21 = 3213212121$$

5. A clock has an hour hand of length 2 and a minute hand of length 4. From 3:00 am to 3:00 pm of the same day, find the number of occurrences when the distance between the tips of the two hands is an integer.

Solution

The shortest distance is 2, and longest distance is 6. So, from 3am to 3pm, there are 12 hours, the short hand and the long hand overlap (i.e. distance = 2) for 11 times. Between any two overlapping, distance equals to 3, 4, 5 occurred twice and distance equals to 6 occurred once, i.e. total 7 times in an hour. Total $7 \times 10 = 70$. For the first hour before overlap and last hour after overlap, we have another 7 times integer distance. So in total $11 + 7 \times 10 + 7 = 88$.