BRITISH COLUMBIA SECONDARY SCHOOL MATHEMATICS CONTEST, 2025

Senior Final, Part B Problems & Solutions

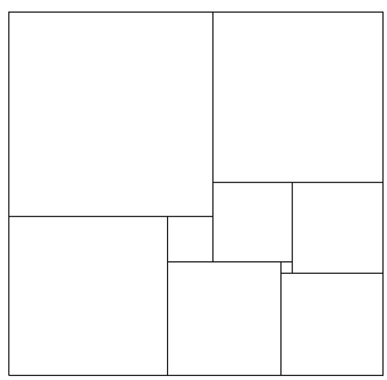
1. For some integers x and y, the number $3x + 2y$ is divisible by 23. Prove that the number $17x + 19y$ is also divisible by 23.

   Solution

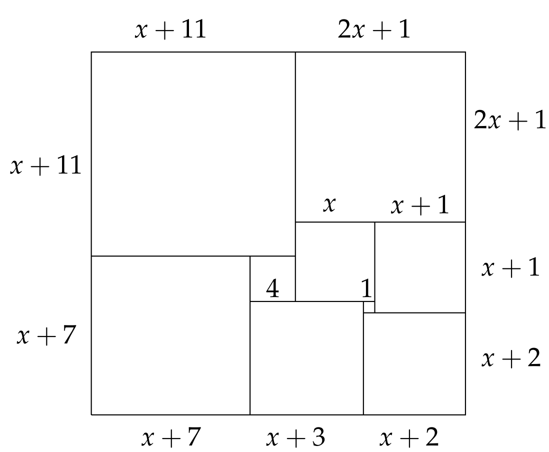
   Note that
   $$17x + 19y + 2(3x + 2y) = 23x + 23y = 23(x + y),$$
   so, divisible by 23.

   Since $3x + 2y$ is divisible by 23, the difference
   $$23(x + y) - 2(3x + 2y)$$
   which is $17x + 19y$ will be also divisible by 23.

2. The diagram shown is a rectangle that has been divided into 9 squares. If the smallest square has sides of length 1, what are the lengths of the sides of the rectangle?



   Solution



$$x + 11 + x + 7 = 2x + 1 + x + 1 + x + 2 \implies x = 7.$$

The rectangle's sides are 33 and 32.

3. In a six-digit number, the first digit is the same as the fourth digit, the second digit is the same as the fifth digit and the third digit is the same as the sixth digit. Prove that the number is divisible by 7, 11 and 13.

Solution

The number n can be expressed as $abcabc$ where a, b and c are any digits with $a \neq 0$. We obtain

$$\begin{aligned} n &= abcabc \\ &= 100000a + 100a + 10000b + 10b + 1000c + c \\ &= 100100a + 10010b + 1001c. \end{aligned}$$

Since each of the numbers 100100, 10010 and 1001 is divisible by 7, 11 and 13, so is the number n.

4. A highly composite number is a positive integer that has more divisors than any smaller positive integer. For example, 6 is a highly composite number because it has more divisors than 1,2,3,4 or 5. What is the smallest highly composite number that is larger than 70?

Solution

Note that if the prime factorization of a positive integer n is given by $n = p_1^{a_1} \cdot p_2^{a_2} \cdots p_k^{a_k}$, then n has exactly $(a_1 + 1)(a_2 + 1) \cdots (a_k + 1)$ positive divisors. We may hence use the prime factorization of any integer to determine how many divisors it has. The following tables display the prime factorizations, and number of positive divisors of the first 120 positive integers:

n	fact	# div
1	N/A	1
2	prime	2
3	prime	2
4	2^2	3
5	prime	2
6	$2 \cdot 3$	4
7	prime	2
8	2^3	4
9	3^2	3
10	$2 \cdot 5$	4
11	prime	2
12	$2^2 \cdot 3$	6
13	prime	2
14	$2 \cdot 7$	4
15	$3 \cdot 5$	4
16	$2 \cdot 4$	5
17	prime	2
18	$2 \cdot 3^2$	6
19	prime	2
20	$2^2 \cdot 5$	6
21	$3 \cdot 7$	4
22	$2 \cdot 11$	4
23	prime	2
24	$2^3 \cdot 3$	8
25	5^2	3
26	$2 \cdot 13$	4
27	3^3	4
28	$2^2 \cdot 7$	6
29	prime	2
30	$2 \cdot 3 \cdot 5$	8

n	fact	# div
31	prime	2
32	2^5	6
33	$3 \cdot 11$	4
34	$2 \cdot 17$	4
35	$3 \cdot 5$	4
36	$2^2 \cdot 3^2$	9
37	prime	2
38	$2 \cdot 19$	4
39	$3 \cdot 13$	4
40	$2^3 \cdot 5$	8
41	prime	2
42	$2 \cdot 3 \cdot 7$	8
43	prime	2
44	$2^2 \cdot 11$	6
45	$3^2 \cdot 5$	6
46	$2 \cdot 23$	4
47	prime	2
48	$2^4 \cdot 3$	10
49	7^2	3
50	$2 \cdot 5^2$	6
51	$3 \cdot 17$	4
52	$2^2 \cdot 13$	6
53	prime	2
54	$2 \cdot 3^3$	8
55	$5 \cdot 11$	4
56	$2^3 \cdot 7$	8
57	$3 \cdot 19$	4
58	$2 \cdot 29$	4
59	prime	2
60	$2^2 \cdot 3 \cdot 5$	12

n	fact	# div
61	61	2
62	$2 \cdot 31$	4
63	$3^2 \cdot 7$	6
64	2^6	7
65	$5 \cdot 13$	4
66	$2 \cdot 3 \cdot 11$	8
67	prime	2
68	$2 \cdot 39$	4
69	$3 \cdot 23$	4
70	$2 \cdot 5 \cdot 7$	8
71	prime	2
72	$2^3 \cdot 3^2$	12
73	prime	2
74	$2 \cdot 37$	4
75	$3 \cdot 5^2$	6
76	$2^2 \cdot 19$	6
77	$7 \cdot 11$	4
78	$2 \cdot 3 \cdot 13$	8
79	prime	2
80	$2^4 \cdot 5$	10
81	3^4	5
82	$2 \cdot 41$	4
83	prime	2
84	$2^2 \cdot 3 \cdot 7$	12
85	$5 \cdot 17$	4
86	$2 \cdot 43$	4
87	$3 \cdot 29$	4
88	$2^3 \cdot 11$	8
89	prime	2
90	$2 \cdot 3^2 \cdot 5$	12

n	fact	# div
91	$7 \cdot 13$	4
92	$2^2 \cdot 23$	6
93	$3 \cdot 31$	4
94	$2 \cdot 47$	4
95	$5 \cdot 19$	4
96	$2^5 \cdot 3$	12
97	prime	2
98	$2 \cdot 7^2$	6
99	$3^2 \cdot 11$	6
100	$2^2 \cdot 5^2$	9
101	prime	2
102	$2 \cdot 3 \cdot 17$	8
103	prime	2
104	$2^3 \cdot 13$	8
105	$3 \cdot 5 \cdot 7$	8
106	$2 \cdot 53$	4
107	prime	2
108	$2^2 \cdot 3^3$	12
109	prime	2
110	$2 \cdot 5 \cdot 11$	8
111	$3 \cdot 37$	4
112	$2^4 \cdot 7$	10
113	prime	2
114	$2 \cdot 3 \cdot 19$	8
115	$5 \cdot 23$	4
116	$2^2 \cdot 29$	6
117	$3^2 \cdot 11$	6
118	$2 \cdot 59$	4
119	$7 \cdot 17$	4
120	$2^3 \cdot 3 \cdot 5$	16

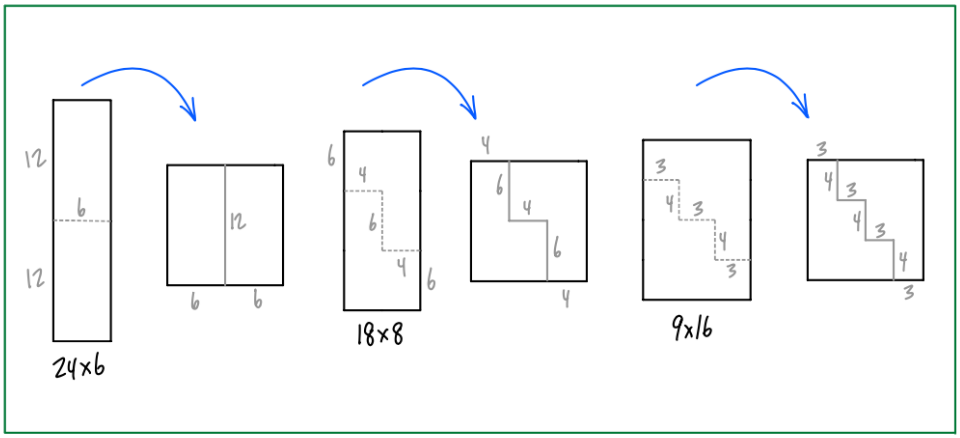
From this table it may be seen that $1, 2, 4, 6, 12, 24, 36, 48, 60$, and 120 are highly composite. Hence the answer is 120.

5. Find the dimensions of all rectangles with the following properties:

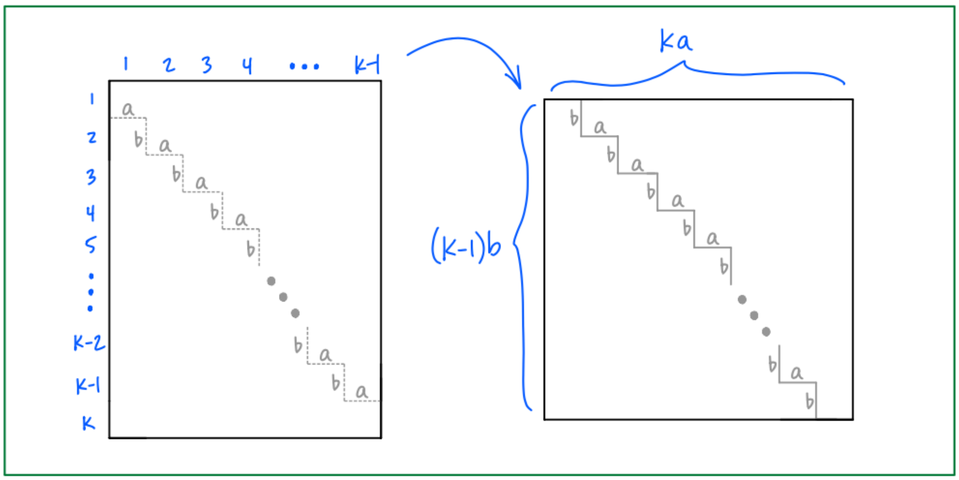
   1. Not a square
   2. Integer side lengths
   3. Using piecewise straight cuts, it can be decomposed into two pieces which can be reassembled into a 12 by 12 square.

Solution

A 24×6 rectangle may be cut as desired with a single straight cut. Since the cuts are not required to be a straight line, 18×8 or 9×16 rectangles may also be cut as desired in a zig zagging fashion.



To see that these are the only suitable rectangles, consider a generalized zig zagging cut into a rectangle with $2k$ teeth with area $a \times b$.



The original rectangle must have dimensions $kb \times (k-1)a$. Since it must have integer side-lengths, both a and b must be integers. Moreover, since $ka = 12$ and $(k-1)b = 12$, both k and $k-1$ must be consecutive divisors of 12. Since 12 has only the divisors $1, 2, 3, 4, 6, 12$, we must have $k = 1, 2$, or 3. These values of k yield precisely the three examples above.