BRITISH COLUMBIA SECONDARY SCHOOL MATHEMATICS CONTEST, 2025

Senior Final, Part A Problems & Solutions

1. If $x + y + z = 1$ and $x^2 + y^2 + z^2 = 3$, then the value of $xy + xz + yz$ is:

   (A) 1   (**B**$*$) -1 (C) 2   (D) -2 (E) 4

   Solution

   Since $x + y + z = 1$, we must have also have $(x + y + z)^2 = 1$. So

$$\begin{aligned}
1 &= (x + y + z)^2 \\
&= x^2 + xy + xz + yx + y^2 + yz + zx + zy + z^2 \\
&= x^2 + y^2 + z^2 + 2(xy + zy + yz) \\
&= 3 + 2(xy + zy + yz).
\end{aligned}$$

   Therefore $xy + zy + yz = \frac{1-3}{2} = -1$ and the correct answer is (B).

   Answer: B

2. Nicole and Adrienne just became friends with Harmony, and they want to know when her birthday is. Harmony gives them a list of twelve possible dates:

   March 1, 26
   May 18, 28
   June 16, 18, 20
   September 1, 16, 28
   November 3, 18

   Harmony then tells Nicole the month and Adrienne the day of her birthday. Nicole and Adrienne then have the following conversation:

   Nicole: I don't know when Harmony's birthday is, but I know that Adrienne doesn't know either.

   Adrienne: At first I didn't know when Harmony's birthday is, but I know now.

   Nicole: I STILL DON'T know when Harmony's birthday is.

   What month is Harmony's birthday?

   (A) March   (B) May   (C) June   (**D**$*$) September   (E) November

   Solution

   For Adrienne not to know the birthday immediately, the number could not be any of $3, 20$ or 26. Since Nicole knew this, she must have been told that the month was either May or September. Adrienne also arrives at this conclusion, and somehow determines the month. This tells Nicole that the birthday must be September 1st, September 16th, or March 28th. Since she still does not know the birthday, it must September 1st or 16th and the correct answer is (D)

Day repeated	March, September May, June, November September, May June, September	1,1 18, 18, 18 28, 28 16, 16
Day NOT repeated	March June November	26 20 3

Answer: D

3. The largest prime factor of $3^{14} + 3^{13} - 12$ is:

(A) 3 (B) 7 (C) 13 (D) 29 (**E**∗) 73

Solution

The prime factorization of $3^{14} + 3^{13} - 12$ may be computed as

$$\begin{aligned}
3^{14} + 3^{13} - 12 &= 3(3^{13} + 3^{12} - 4) \\
&= 3(3^{12} \cdot 4 - 4) \\
&= 3 \cdot 4 \cdot (3^{12} - 1) \\
&= 2^2 \cdot 3 \cdot (3^6 - 1)(3^6 + 1) \\
&= 2^2 \cdot 3 \cdot (3^3 - 1)(3^3 + 1)(3^6 + 1) \\
&= 2^2 \cdot 3 \cdot 26 \cdot 28 \cdot (3^6 + 1) \\
&= 2^5 \cdot 3 \cdot 7 \cdot 13 \cdot (3^6 + 1) \\
&= 2^5 \cdot 3 \cdot 7 \cdot 13 \cdot 730 \\
&= 2^6 \cdot 3 \cdot 5 \cdot 7 \cdot 13 \cdot 73
\end{aligned}$$

Hence 73 is the largest prime divisor and the correct answer is (E).

Answer: E

4. A function f is defined as follows

$$f(x) = \frac{ax + b}{cx + d}$$

where a, b, c and d are constants. Suppose that $f(0) = 1$, $f(1) = 0$ and $f(2) = 3$. Find $f(3)$.

(A) 0 (B) 1 (**C**∗) 2 (D) 7 (E) 10

Solution

Since $f(0) = \dfrac{b}{d} = 1$, we get $b = d$, and so

$$f(x) = \frac{ax + b}{cx + b}.$$

Since $f(1) = \dfrac{a+b}{c+b} = 0$, we get $b = -a$, and so

$$f(x) = \frac{ax-a}{cx-a}.$$

Similarly, since $f(2) = \dfrac{2a-a}{2c-a} = 3$, we get $a = 6c - 3a$. Then $c = \dfrac{2}{3}a$. We represent $f(x)$ as

$$f(x) = \frac{ax-a}{\frac{2}{3}ax-a} = \frac{x-1}{\frac{2}{3}x-1} = \frac{3(x-1)}{2x-3}.$$

Note that a cannot be 0. Indeed, if $a = 0$, we obtain

$$1 = \frac{b}{d}, \quad 0 = \frac{b}{c+d}, \quad \text{and} \quad 3 = \frac{b}{3c+d}$$

which cannot hold simultaneously. Therefore,

$$f(3) = \frac{3(3-1)}{2\cdot 3-3} = 2.$$

Answer: C

5. Two cyclists are k miles apart and start cycling at the same time. If they cycle in the same direction, the faster cyclist will pass the slower cyclist in r hours. If they cycle towards each other, they will pass each other in t hours. The ratio of the speed of the faster cyclist to the speed of the slower cyclist is

(**A**∗) $\dfrac{r+t}{r-t}$ (B) $\dfrac{r}{r-t}$ (C) $\dfrac{r+t}{r}$ (D) $\dfrac{r}{t}$ (E) $\dfrac{r+k}{t-k}$

Solution

Let s_1 and s_2 be the speeds of the faster and slower cyclist, respectively. It is given that

$$s_1 + s_2 = k/t \text{ and } s_1 - s_2 = r/t.$$

Adding and subtracting these equations yields

$$2s_1 = \frac{k}{t} + \frac{r}{t} \text{ and } 2s_2 = \frac{k}{t} - \frac{r}{t}.$$

Hence

$$\frac{s_1}{s_2} = \frac{2s_1}{2s_2} = \frac{\frac{k}{t}+\frac{r}{t}}{\frac{k}{t}-\frac{r}{t}} = \frac{r+t}{r-t},$$

making (A) the correct answer.

Answer: A

6. A deck of cards contains three red, six green and k blue cards ($k \geq 1$). If two cards are drawn at random without replacement, you are equally likely to draw two cards of the same colour as two different colours. Calculate k.

(A) 5 (B) 7 (C) 13 (D) 15 (**E**∗) 19

Solution

For $j = 1, 2$, let each of R_j, G_j and B_j respectively denote the events that a red, green, or ball is chosen for the first and second drawn card. The probability that both cards are the same colour is given by

$$\frac{3}{9+k}\cdot\frac{2}{8+k}+\frac{6}{9+k}\cdot\frac{5}{9+k}+\frac{k}{9+k}\cdot\frac{k-1}{8+k}=\frac{36-k+k^2}{(9+k)(8+k)}.$$

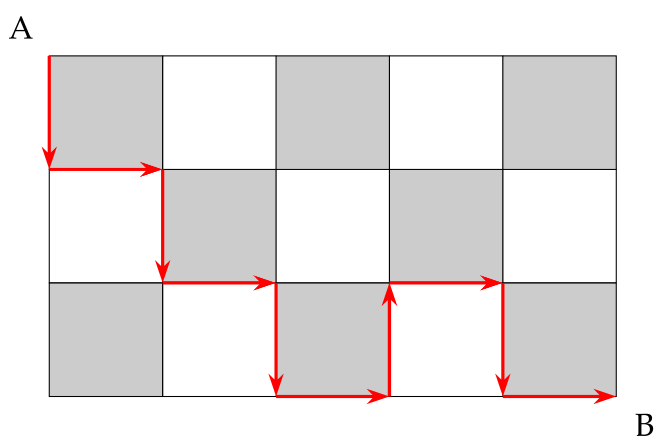
The probability that each card is a different colour is given by

$$\frac{3}{9+k}\cdot\frac{6+k}{8+k}+\frac{6}{9+k}\cdot\frac{3+k}{8+k}+\frac{k}{9+k}\cdot\frac{9}{8+k}=\frac{36+18k}{(9+k)(8+k)}.$$

By assumption, these probabilities are equal, which means that $36+18k=36-k+k^2$, which is true when $k=0$ or $k=19$. Since there must be blue cards, $k=19$. The answer is (E).

Answer: E

7. Fifteen tiles are arranged as shown. An ant walks along the edges of the tiles, always keeping a black tile on its left. The ant never traverses the same edge twice.

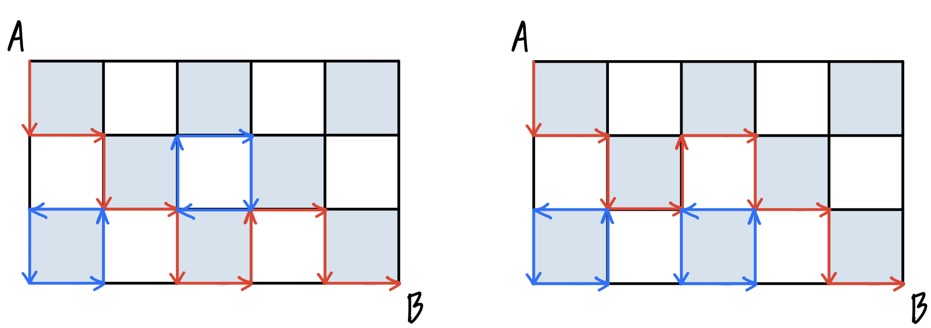


How many different routes could the ant take to get from A to B?

(**A**∗) 8 (B) 4 (C) 2 (D) 10 (E) 6

Solution

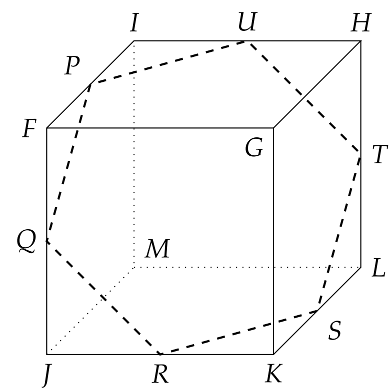
There are two distinct *paths* from A to B (shown below in red), each of which may be extended with two optional loops (shown below in blue).



To avoid double-counting, insist that blue edges are traversed with priority. Considering each blue loop as optional, there are four possible variations of each path, and hence 8 possibilities in total.

Answer: A

8. Given a cube (with corners $FGHIMJKL$), mid-points ($PQRSTU$) of six of the twelve edges of the cube are joined to form a regular hexagon. Find the ratio of the area of the hexagon to the surface area of the cube.



(A) $\sqrt{2} : 4$ (B) $\sqrt{2} : 8$ (C) $\sqrt{3} : 4$ **(D∗)** $\sqrt{3} : 8$ (E) $\sqrt{6} : 12$

Solution

Suppose the cube has centre $(0, 0, 0)$ and that edges are parallel with the axes. Then all of the points P, Q, R, S, T, and U lie in the plane given by $x + y + z = 0$, so the hexagon lies in this plane, and the centre of the hexagon is the centre of the cube.

If the cube has side $2x$, then every side of the hexagon has length $\sqrt{2}x$. Moreover, the distance from any vertex of the hexagon to the centre is also $\sqrt{2}x$. The area of the hexagon is therefore

$$6 \times \tfrac{1}{2}(\sqrt{2}x)^2 \sin(\pi/3) = 6 \times \tfrac{1}{2}2x^2 \times \sqrt{3}/2 = 3\sqrt{3}x^2.$$

The area of the cube is $6(2x)^2 = 24x^2$, so the ratio is

$$3\sqrt{3}x^2 : 24x^2 = \sqrt{3} : 8$$

$\boxed{\textbf{Answer: D}}$

9. In Canada's Got Talent, three judges vote on four finalists: Kathryn, Mei-ling, Nicolas, and Pardeep. The judges rank the finalists without ties. In how many ways can the judges rank the finalists so that two of the judges agree in their order of preference while the third differs? Note: For any natural number n, $n! = 1 \times 2 \times 3 \times \cdots \times n$.

(A) $3! \times 4!$ (B) $3 \times 24!$ (C) $3! \times 23$ **(D∗)** $\dfrac{3 \times 24!}{22!}$ (E) $3! \times 23 \times 24$

Solution

Suppose judge 1 and judge 2 have the same order of preference and judge 3 does not. There are 4! ways for judges 1 and 2 to select their order and so there are $4! - 1$ ways for judge 3 to have another opinion. For this configuration of judges preferences (judge 1 and 2 agree, and judge 3 does not) we have $(4!) \cdot (4! - 1)$ ways. To find the total number one must triple the last value. The answer is $3 \cdot (4!) \cdot (4! - 1) = 3 \cdot 23 \cdot 24$. The answer is (D).

$\boxed{\textbf{Answer: D}}$

10. A deck of 16 cards contains 4 Jacks, 4 Queens, 4 Kings, 4 Aces. I shuffle the deck and draw 2 cards at random. I then tell you (truthfully) that I have at least one ace. What is the probability that I have two aces?

    (A) $\frac{1}{5}$ (B) $\frac{1}{6}$ (**C***) $\frac{1}{9}$ (D) $\frac{3}{16}$ (E) $\frac{2}{15}$

    Solution

    Consider the set of all ordered pairs of cards with at least one ace in each. For example, $(\text{Ace}\,\heartsuit, \text{Jack}\,\spadesuit)$, $(\text{King}\,\clubsuit, \text{Ace}\,\diamondsuit)$ or $(\text{Ace}\,\clubsuit, \text{Ace}\,\spadesuit)$. Let us count all such pairs. There are $4 \times 12 \times 2 = 96$ pairs with exactly one ace and there are $4 \times 3 = 12$ of pairs with two aces. So, the total number of such pairs is $96 + 12 = 108$. Then the probability of having two aces given that there is at least one ace equals $\frac{12}{108} = \frac{1}{9}$. The answer is (C).

    $\boxed{\textbf{Answer: C}}$