BRITISH COLUMBIA SECONDARY SCHOOL MATHEMATICS CONTEST, 2024

Senior Preliminary Problems & Solutions

1. The smallest positive integer having the remainders 2, 3 and 6 when divided by 3, 5, and 11, respectively, lies between:

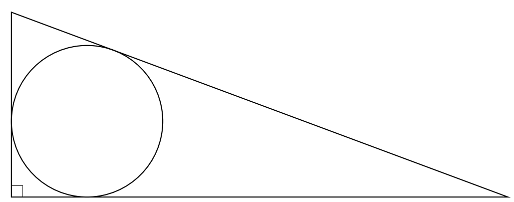
   (A) 41 and 50 (B) 61 and 70 (C) 71 and 80 (**D**∗) 81 and 90 (E) 91 and 100

   Solution

   The answer is 83. The easiest way to find it is by finding numbers with remainder 6 when divided by 11: 6, 17, 28, 39, 50, 61, 72, 83, 94. Of these, only 28 and 83 have remainder 3 when divided by 5. Checking those, 83 is the only one that has remainder 2 when divided by 3.

   Answer: D

2. The hypotenuse of a right triangle is 10 cm and the radius of the inscribed circle is 1 cm.

   The perimeter of the triangle in centimeters is:

   (A) 16 (**B**∗) 22 (C) 23 (D) 24 (E) 26

   Solution

   Use the tangent property of a circle to get $2 \times 10 + 2 = 22$

   Answer: B

3. If $y = f(x) = \frac{x+2}{x-1}$, then it is incorrect to say:

   (A) $x = \frac{y+2}{y-1}$ (B) $f(0) = -2$ (**C**∗) $f(1) = 0$ (D) $f(-2) = 0$ (E) $f(y) = x$

   Solution

   Note that if we do the algebra to find the inverse of $f(x)$ we see that $f(x)$ is its own inverse, making (A) and (E) both true. Plugging in $x = 1$ we get 0 in the denominator, so $f(1)$ is undefined, not 0. It can be verified that (B) and (D) both work.

   Answer: C

4. A 25-foot ladder is placed against a wall of a building. The bottom of the ladder is 7 feet from the base of the building. If the top of the ladder slides down 4 feet, then the foot of the ladder will slide:

(A) 9 feet (B) 15 feet (C) 5 feet (**D**$*$) 8 feet (E) 4 feet

Solution

We have a right triangle with the length of the ladder as the hypotenuse, and the height of the top of the ladder, and the distance between the bottom of the ladder and the wall, as legs. Using the Pythagorean Theorem, the original height of the top of the ladder is $\sqrt{25^2 - 7^2} = 24$ feet. If the top of the ladder slides down four feet to 20 feet, the bottom of the ladder will now be $\sqrt{25^2 - 20^2} = 15$ feet away from the wall, so it will have moved 8 feet.

Answer: D

5. A rectangular box has side, front, and bottom faces with areas of $70\sqrt{3}\,\text{cm}^2$, $42\sqrt{2}\,\text{cm}^2$, and $10\sqrt{6}\,\text{cm}^2$ respectively. The volume of the box is:

(**A**$*$) $420\,\text{cm}^3$ (B) $420\sqrt{6}\,\text{cm}^3$ (C) $840\,\text{cm}^3$ (D) $420\sqrt{3}\,\text{cm}^3$ (E) $420\sqrt{2}\,\text{cm}^3$

Solution

Let side lengths of the rectangular box be l, w, and h. WLOG, assume that $lw = 70\sqrt{3}$, $lh = 42\sqrt{2}$, and $wh = 10\sqrt{6}$. If we multiply the three equations, we get $(lwh)^2 = 6^2 \times 7^2 \times 10^2$, so $V = lwh = 420$

Answer: A

6. All possible five-letter "words" that can be made with the letters a, b, c, d, and e are put in alphabetical order: aaaaa, aaaab, aaaac, etc. The 2024$^{\text{th}}$ "word" will be:

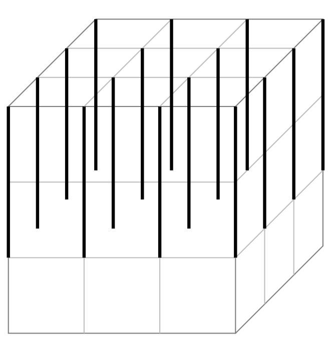
(A) dabed (B) daebd (C) dbbae (**D**$*$) dbaed (E) dbaee

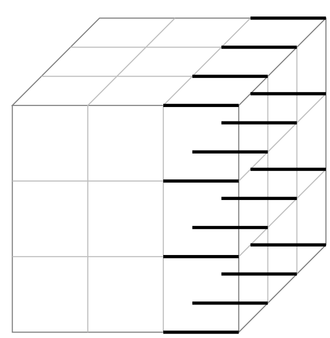
Solution

With 5 letters available, there are $5^5 = 3125$ strings. Since $5^4 = 625$, and $3 \times 625 = 1875$, the first letter must be a "d". Using a similar argument, the string "daeee" is the 2000th string. Thus, "dbaed" is the 2024th.

Answer: D

7. Consider a $3 \times 3 \times 3$ cube composed of 27 unit cubes as an example. When an impatient painter paints the top face, paint seeps along the edges down two levels. When the same painter paints the side faces (right hand face here), paint seeps along the edges to only one level as shown in the diagrams.





Suppose the same impatient painter paints the top face orange, the right and left faces red, and the front and back faces blue on a $5 \times 5 \times 5$ cube. (The bottom face is not accessible, so never painted.) How many unit cubes with exactly two faces painted have their edges painted by all three colours?

(A) 25 (B) 16 (C) 8 (D) 5 (**E**∗) 4

Solution

Answer: E

8. If $f(x) = x + 1$ and $F(x, y) = y^2 + x$ then $F(2, f(3)) =$

(A) $x^2 + 3x + 1$ (B) 19 (**C**∗) 18 (D) 8 (E) 4

Solution

First we find $f(3) = 3 + 1 = 4$. Then we find $F(2, 4) = 4^2 + 2 = 16 + 2 = 18$.

Answer: C

9. Some time after school has ended, there are a group of students and teachers still in the gym. First, fifteen students leave, then there are two teachers per student. Then 45 teachers leave, and there are 5 students per teacher. Find the number of students in the gym at the beginning.

(A) 50 (B) 48 (C) 45 (D) 43 (**E**∗) 40

Solution

Let S be the original number of students and T be the original number of teachers. After 15 students leave, we know: $2(S - 15) = T$ so $T = 2S - 30$. After 45 teachers leave, we have $S - 15 = 5(T - 45) = 5(2S - 30 - 45) = 10S - 375$. Solving the equation, we get $S = 40$.

Answer: E

10. When riding on a train, a person counts posts that are spaced 10 meters apart alongside the tracks. For how many seconds must a person count posts in order that the number of posts counted matches exactly the numerical value of the speed of the train in km per hour?

(A) 100 (B) 60 (C) 45 (**D**∗) 36 (E) 24

Solution

Let the speed of the train be s kilometers per hour. Then it is equivalent to $1000s$ metres in 3600 seconds, or $10s/36$. Therefore, by counting clicks for 36 seconds, the speed of the train in km/h will match the number of posts they are spaced 10 metres apart.

Answer: D

11. The digits 1, 2, 3, 4, and 5 are each used once to compose a five-digit number *abcde*, such that the three digit number *abc* is divisible by 4, *bcd* is divisible by 5, and *cde* is divisible by 3. The digit *a* is:

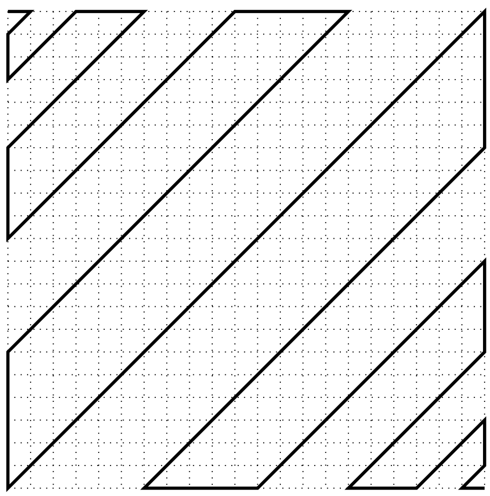
(**A**∗) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution

Since abc is divisible by 4, the last digit, c, must be 2 or 4. Since bcd is divisible by 5, the last digit, d, must be 5. Since cde is divisible by 3, the sum of the digits, $c + d + e$, must be divisible by 3. We know $d = 5$ and $c = 2$or $c = 4$. If $c = 2$, then there is no remaining available digit for e that works, so $c = 4$. Then $e = 3$ is the only value that works, and $a = 1$.

Answer: A

12. Shown is a 21×21 square with diagonal zigzags alternating with horizontal or vertical segments beginning with lengths 1 (horizontal), 2 (vertical), 3 (horizontal), 4 (vertical), etc. Using the same pattern for a 105×105 square, find the length of the dark line that starts at the upper left hand corner and ends at the lower right hand corner.



(A) $210 + 1120\sqrt{2}$ (B) $210 + 1105\sqrt{2}$ (C) $210 + 1100\sqrt{2}$ **(D∗)** $210 + 1015\sqrt{2}$ (E) $210 + 560\sqrt{2}$

Solution

The horizontal and vertical line segments match exactly two sides of the square, so $2 \times 105 = 210$. The diagonal line segments are triangular numbers; since the sum of consecutive triangular numbers is a square, $1 + 3 + 6 + 10 + \cdots + 91 + 105 + 91 + 78 + \cdots + 10 + 6 + 3 + 1 = 2^2 + 4^2 + 6^2 + \cdots + 14^2 + 13^2 + 11^2 + \cdots + 1^2 = 1015$ unit diagonals, so $1015\sqrt{2}$.

Answer: D