BRITISH COLUMBIA SECONDARY SCHOOL MATHEMATICS CONTEST, 2024

Senior Final, Part B Problems & Solutions

1. Find the smallest number such that the sum of the cubes of its digits is not divisible by the sum of its digits. Explain.

Solution

For any single-digit number, x, clearly x^3 is divisible by x.

Note that $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$. (This is easily verified by expanding the right-hand side.) Therefore $x^3 + y^3$ is always divisible by $x + y$. For any two-digit number with digits x and y, the sum of the cubes of the digits is therefore divisible by the sum of the digits.

For the numbers $100, 101, \ldots, 109, 110$, let x and y be the non-zero digits. Then the sum of the cubes of the digits is $x^3 + y^3$ and the sum of the digits is $x + y$, so, again, the cube sum is divisible by the digit sum.

For 111 the cube sum is $1^3 + 1^3 + 1^3 = 3$ and the digit sum is also 3, so one is divisible by the other.

However, for 112, $1^3 + 1^3 + 2^3 = 10$ and $1 + 1 + 2 = 4$, but 10 is not divisible by 4. Therefore 112 is the smallest such number.

2. Determine the number of positive integral solutions of the equation $a^2 - 7a + b^2 - 7b + 2ab = 0$.

Solution

Note that $a^2 - 7a + b^2 - 7b + 2ab = a^2 + 2ab + b^2 - 7a - 7b = (a + b)^2 - 7(a + b) = (a + b)(a + b - 7)$. Therefore $a^2 - 7a + b^2 - 7b + 2ab = 0$ when $a + b = 0$ or $a + b - 7 = 0$. Since $a > 0$ and $b > 0$, $a + b > 0$ and the only six solutions are

$a =$	1	2	3	4	5	6
$b =$	6	5	4	3	2	1

3. Let f be a real-valued function such that $f(m + n) = f(m)f(n)$. If f(4)=256 and f(k)= 0.0625, find the value of k.

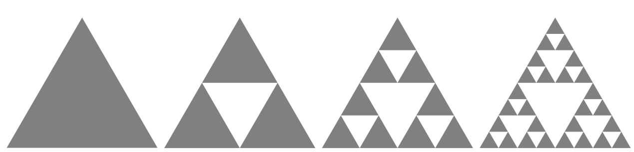
Solution

If both $m = 2$ and $n = 2$, the given equation yields that $256 = f(4) = \big(f(2)\big)^2$, so $f(2) = \sqrt{256} = 16$.

If $n = 0$, then the given equation yields that $f(m) = f(m + 0) = f(0)f(m)$, so $f(0) = 1$.

If $m = 2$ and $n = -2$, then $1 = f(0) = f(2 - 2) = f(2)f(-2) = 16f(-2)$, so $f(-2) = \frac{1}{256} = 0.0625$.

4. The Sierpinski triangle is a self-similar fractal. It is obtained as an equilateral triangle with repeatedly removed smaller middle equilateral triangles from the remaining area. We start with a shaded equilateral triangle (stage 0), then remove the middle triangle (stage 1). For each of the three remained shaded triangles we remove the corresponding middle triangles (stage 2) and so on. The figure below shows the first three stages following stage 0.



a) Find an expression (in terms of n) for s, the number of shaded triangles for stage n.

b) Find an expression (in terms of n) for u, the number of unshaded triangles for stage n.

c) Let $d = s - u$ be the difference between the number of shaded triangles and the number of unshaded triangles for each given stage n. Find and simplify d.

Solution

a) The number of shaded triangles triples at each stage, so $s(n) = 3^n$.

b) At each stage, the triangles that were white before remain white, while each triangle that was shaded before generates one new white triangle. That is

$$u(n) = u(n-1) + s(n-1)$$

But that means that $u(n) = s(n-1) + s(n-2) + \cdots + u(0) = 3^{n-1} + 3^{n-2} + \cdots + 1$. But that's a geometric series, so

$$u(n) = \frac{3^n - 1}{3 - 1} = \frac{3^n - 1}{2}$$

c)

$$d(n) = s(n) - u(n) = 3^n - \frac{3^n - 1}{2} = \frac{3^n + 1}{2}$$

5. The Main Street Math Symposium is a club that has more than one committee. Suppose that

- each committee consists of 4 members from the club.
- every pair of club members serves on exactly one committee together, and
- each pair of committees has at least one member in common.

a) Show that every two committees have exactly one member in common.

b) Show that each person is on at least 4 committees.

c) Show that each person is on at most 4 committees.

d) How many members of the club are there?

Solution

a) We are given that each pair of committees has at least one member in common. Suppose a pair of committees has two members in common. Then those two members serve on (at least) two committees together. This contradicts the given fact that every pair of members serves on exactly one committee together. Therefore every pair of committees have at most (and, therefore, exactly) one member in common.

b) Pick a committee at random and name the members A, B, C, and D. There are more than one committee, and the second committee cannot consist of the same 4 members, so it must have a new member, who we'll call E. We know E will need to be on a committee with each of A, B, C, and D, or at least 4 committees.

c) Suppose E is on a fifth committee. They are already serving on 4 committees, one with each of A, B, C, and D. Therefore, the fifth committee would not be able to contain A, B, C, or D. This is a contradiction, because the new committee would have no member in common with our original committee.

d) Let M = the number of members of the club Let C = the number of committees Then the number of committee spots total can be expressed as 4M or 4C, so M=C.

The number of pairs of committees (or members) is C(C-1)/2, which we could think of alternately as 6 pairs of members in each committee, times C committees. Therefore $C(C-1)/2 = 6C$ which we can solve to get $C = M = 13$

Alternate solution for d) A, B, C, and D are on one committee all together, and exactly three other committees each, completely separately. The number of committees is $1 + 4(3) = 13$

We can also find committees that work:

ABCD AEFG AHJL AIKM

BEHI BFJM BGKM

CEJK CFLI CGHM

DELM DFHK DGJI

Note there are 13 committees and 13 distinct members.