BRITISH COLUMBIA SECONDARY SCHOOL MATHEMATICS CONTEST, 2024

Senior Final, Part A Problems & Solutions

1. If $f(n)$ is the minimum value of $f(x) = 2^{x^2-6x}$, then the value of $\frac{n}{f(n)}$ is:

   (A) 0 (B) 1 (C) 8 (D) -8 (E) 3×2^9

   Solution

   Since $x^2 - 6x = (x-3)^2 - 9$, the minimal value of $x^2 - 6x$ is obtained when $x = 3$. The minimal value of $f(x)$ is therefore obtained when $x = 3$, so $n = 3$. Now

   $$\frac{n}{f(n)} = \frac{3}{f(3)} = \frac{3}{2^{-9}} = 3 \times 2^9.$$

   $\boxed{\textbf{Answer: E}}$

2. A trombone has been broken! Three students are being questioned by their music teacher. They each make comments about each other. Anglin says, "Ethan is lying." Ethan says, "Xavier is lying." Xavier says "both Anglin and Ethan are lying." Who is lying?:

   (A) only Anglin (B) only Ethan (C) only Xavier (D) both Anglin and Ethan

   (E) both Anglin and Xavier

   Solution

   Suppose Anglin is telling the truth. Then Ethan must be lying, but if his statement is false, Xavier must also be telling the truth. In that case, Xavier's statement contradicts our original assumption that Anglin was telling the truth. Therefore, our assumption that Anglin is telling the truth can't be true, so Anglin must be lying. That means Ethan is telling the truth, and Xavier is also lying.

   $\boxed{\textbf{Answer: E}}$

3. Which of the following is a divisor of $x^{17} - 4x^{15} - x^3 + 8$?

   (A) $x+3$ (B) $x+2$ (C) $x-2$ (D) $x+1$ (E) $x-1$

   Solution

   Since $x = 2$ is a root of $x^{17} - 4x^{15} - x^3 + 8$ then $x - 2$ must be a divisor. Note that $x = -3$, $x = -2$, $x = -1$, $x = 1$ are not roots.

   $\boxed{\textbf{Answer: C}}$

4. If $f(x) = 10^x$, then $f(x+1) - f(x) =$

   (A) 10 (B) 90 (C) $f(x)$ (D) $f(x) - 1$ (E) $9f(x)$

Solution

$f(x+1) - f(x) = 10^{x+1} - 10^x = (10-1)10^x = 9f(x)$.

Answer: E

5. The absolute value of x, written as $|x|$, is the distance of x from 0. For example, $|-5| = 5$, and $|3| = 3$. Determine the number of solutions of $|x||y||z| = 12$, such that x, y, and z are all integers.

   (A) 36 (B) 48 (C) 72 (D) 144 (E) 180

Solution

Twelve can be written as a product of three positive integer factors in 18 ways, namely $1 \times 1 \times 12$, $1 \times 12 \times 1$, $12 \times 1 \times 1$, $1 \times 2 \times 6$, $2 \times 1 \times 6$, $1 \times 6 \times 2$, $2 \times 6 \times 1$, $6 \times 1 \times 2$, $6 \times 2 \times 1$, $1 \times 3 \times 4$, $3 \times 1 \times 4$, $1 \times 4 \times 3$, $3 \times 4 \times 1$, $4 \times 1 \times 3$, $4 \times 3 \times 1$, $2 \times 2 \times 3$, $2 \times 3 \times 2$, and $3 \times 2 \times 2$.

With the absolute value, each factor can by either positive or negative, so there are $2^3 \times 18 = 144$ possibilities.

Answer: D

6. Consider equation (i): $x + y + z = 46$ and equation (ii): $p + q + r + s = 46$.
   Which of the following statements is true?

   (A) equation (i) can be solved with three consecutive integers

   (B) equation (i) can be solved with three consecutive even integers

   (C) equation (ii) can be solved with four consecutive integers

   (D) equation (ii) can be solved with four consecutive even integers

   (E) equation (ii) can be solved with four consecutive odd integers

Solution

Since $14 + 15 + 16 = 45 < 46$ and $15 + 16 + 17 = 48 > 46$, equation (i) cannot be solved by consecutive integers.

Since $12 + 14 + 16 = 42 < 46$ and $14 + 16 + 18 = 48 > 46$, equation (i) cannot be solved by consecutive even integers.

Since $8 + 10 + 12 + 14 = 44 < 46$ and $10 + 12 + 14 + 16 = 52 > 46$, equation (ii) cannot be solved by consecutive even integers.

Since $7 + 9 + 11 + 13 = 40 < 46$ and $9 + 11 + 13 + 15 = 48 > 46$, equation (ii) cannot be solved by consecutive dd integers.

However,

$$10 + 11 + 12 + 13 = 46.$$

Answer: C

7. All of 7 different books are distributed to four people: Maddie, Sabrina, Sacha and Adele. In how many ways can this be done if each person is to receive at least one book and no person is to receive more than two books?

(A) 15120 (B) 10080 (C) 7560 (D) 5040 (E) 2520

Solution

Invent a "ghost book" (maybe a single sheet of paper) to add the the seven books. Then you have eight books, and each person must receive exactly two books. Then Maddie's two books can be chosen in ${}_8C_2 = \frac{8!}{2!\cdot 6!} = 28$ ways, Sabrina's can be chosen in ${}_6C_2 = \frac{6!}{2!\cdot 4!} = 15$ ways, Sacha's in ${}_4C_2 = \frac{4!}{2!\cdot 2!} = 6$ ways, and Adele's in 1 way.

Grand total $28 \times 15 \times 6 \times 1 = 2520$ choices. Then remove the ghost book from whomever has it.

(The arithmetic is maybe easier done as $\frac{8!}{2!\cdot 6!} \cdot \frac{6!}{2!\cdot 4!} \cdot \frac{4!}{2!\cdot 2!} = \frac{8!}{2!\cdot 2!\cdot 2!\cdot 2!} = 2520$.)

Answer: E

8. The sum of the first ten terms of an arithmetic sequence is four times the sum of the first five terms. If the arithmetic sequence is given by

$$a, a+d, a+2d, a+3d, \ldots$$

then $a : d$ is:

(A) 1:2 (B) 2:1 (C) 1:4 (D) 4:1 (E) 1:1

Solution

The sum of the first n terms is $S_n = \left(a + \frac{d}{2}(n-1)\right)n$. Therefore $S_{10} = (a + \frac{9}{2}d)10 = 10a + 45d$ and $S_5 = (a + 2d)5 = 5a + 10d$. (These values for S_{10} and S_5 can also easily be worked out by hand.)

Since $S_{10} = 4S_5$, $10a + 45d = 4(5a + 10d) = 20a + 40d$, so $5d = 10a$, so $d = 2a$. Therefore $a : d = a : 2a = 1 : 2$

Answer: A

9. Integers m and n are chosen at random from the set $\{1, 2, \ldots, 10\}$ (with replacement, i.e. m and n can repeat). Find the probability that m and n satisfy the equation:

$$m^{n+1} + n^m = 2024$$

(A) 1/10 (B) 1/20 (C) 1/25 (D) 1/50 (E) 1/100

Solution

The equation can be satisfied only when m and n are both even or when m and n are both odd because 2024 is even. This reduces the number of pairs (m, n). It is clear that m cannot be 1.

For $m = 2$, and $n = 2$ or $n = 4$ or $n = 6$ or $n = 8$,

$$m^{n+1} + n^m < 2024.$$

For $m = 2$ and $n = 10$,

$$2^{11} + 10^2 = 2148 > 2024.$$

One can check other pairs by gradually changing m and n.

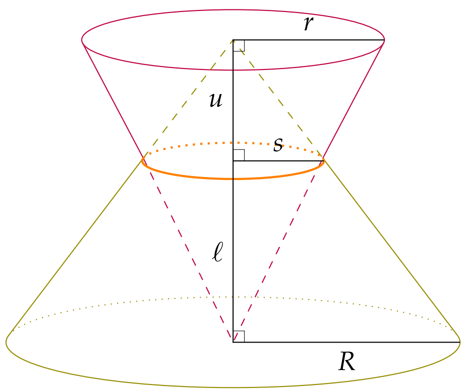
The only pair that satisfies the equation is $(m, n) = (10, 2)$. Since there are 100 ordered pairs in total and the pairs are equally likely to be chosen, the probability is $1/100$.

Answer: E

10. Two right circular cones have the same height but different radii. Their bases lie in parallel planes and one of the cones is inverted so that the vertex of each cone is the center of the base of the other one. The cones intersect at circle C. If the areas of the bases are 400 and 900, then the area of C is:

(A) 100 (B) 144 (C) 288 (D) 350 (E) 650

Solution



Say the smaller cone has radius r, the larger cone has radius R, and the circle C has radius s. Let u be the distance between the planes containing C and the base of the smaller cone, respectively. Similarly, let ℓ be the distance between the planes containing C and the base of the larger cone, respectively.
Then, by similar triangles,

$$\frac{u}{s} = \frac{u+\ell}{R}, \quad \text{and} \quad \frac{\ell}{s} = \frac{u+\ell}{r}.$$

Therefore $\frac{u}{s} + \frac{\ell}{s} = \frac{u+\ell}{R} + \frac{u+\ell}{r}$, so $\frac{u+\ell}{s} = (u+\ell)\left(\frac{1}{R} + \frac{1}{r}\right)$, so $\frac{1}{s} = \frac{1}{R} + \frac{1}{r}$.

Since $\pi r^2 = 400$ and $\pi R^2 = 900$, $r = \frac{20}{\sqrt{\pi}}$ and $R = \frac{30}{\sqrt{\pi}}$. Therefore $\frac{1}{s} = \frac{\sqrt{\pi}}{20} + \frac{\sqrt{\pi}}{30} = \frac{\sqrt{\pi}}{12}$, so $s = \frac{12}{\sqrt{\pi}}$, and the area of C is $\pi s^2 = 144$.

Answer: B