BRITISH COLUMBIA SECONDARY SCHOOL MATHEMATICS CONTEST, 2024

Junior Final, Part B Problems & Solutions

1. In a certain card game, one of the hands dealt contains:

   1. Exactly 13 cards.
   2. At least one card in each suit (hearts, clubs, diamonds, spades)
   3. A different number of cards in each suit.
   4. A total of five hearts and diamonds.
   5. A total of six hearts and spades.
   6. Exactly two cards in the "master" suit.

   Which of the four suits—hearts, clubs, diamonds, or spades—is the “master" suit? Explain.

   Solution

   If the hand has x hearts, then it has $5-x$ diamonds (by rule 4) and $6-x$ spades (by rule 5). The number of clubs is then $13-(x+5-x+6-x)=2+x$ by rule 1. Since there needs to be at least one card in each suit, the possibilities are

$\heartsuit$ x	$\clubsuit$ $2+x$	$\diamondsuit$ $5-x$	$\spadesuit$ $6-x$
1	3	4	5
2	4	3	4
3	5	2	3
4	6	1	2

   The top row does not contain a 2, so it violates rule 6. The next two rows are eliminated by rule 3. Therefore only the last row remains, and the master suit is spades.

2. Find the smallest number such that the sum of the cubes of its digits is not divisible by the sum of its digits. Explain.

   Solution

   For any single-digit number, x, clearly x^3 is divisible by x.

   Note that $x^3+y^3=(x+y)(x^2-xy+y^2)$. (This is easily verified by expanding the right-hand side.) Therefore x^3+y^3 is always divisible by $x+y$. For any two-digit number with digits x and y, the sum of the cubes of the digits is therefore divisible by the sum of the digits.

   For the numbers $100, 101, \ldots, 109, 110$, let x and y be the non-zero digits. Then the sum of the cubes of the digits is x^3+y^3 and the sum of the digits is $x+y$, so, again, the cube sum is divisible by the digit sum.

   For 111 the cube sum is $1^3+1^3+1^3=3$ and the digit sum is also 3, so one is divisible by the other.

   However, for 112, $1^3+1^3+2^3=10$ and $1+1+2=4$, but 10 is not divisible by 4. Therefore 112 is the smallest such number.

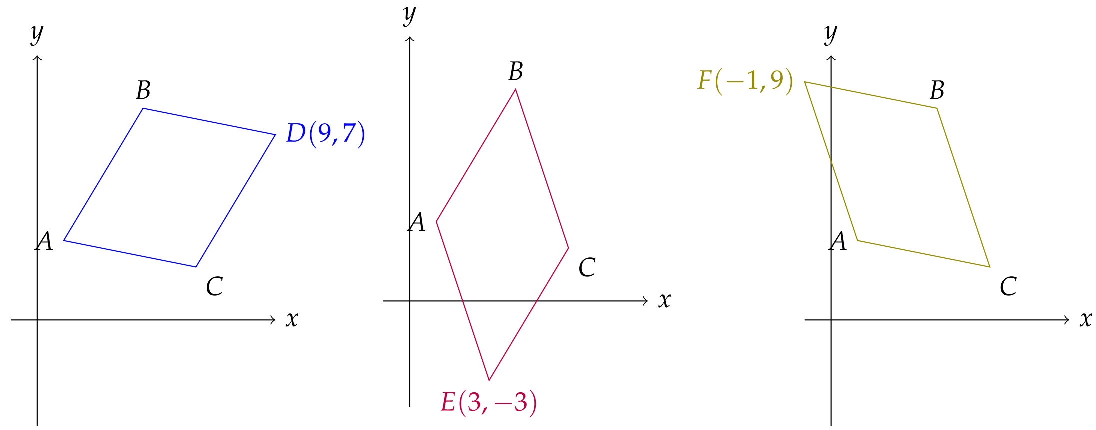
3. Three vertices of a parallelogram in random order are $A(1,3)$, $B(4,8)$, and $C(6,2)$. Find all possible other points that can be the fourth vertex of the parallelogram.

   Solution

   To walk from A to B, go 3 to the right and 5 up. Now walk the same walk from C and end up at $(9,7)=D$. Then $ABDC$ is a parallelogram.

Similarly, to walk from B to C, go 2 right and 6 down. The same walk from A ends at $E = (3, -3)$, yielding the parallelogram $ABCE$.

Finally, to walk from C to A, go 5 left and 1 up. The same walk from B ends at $F = (-1, 9)$ and the parallelogram $AFBC$.



4. The absolute value of x, written as $|x|$, is the distance of x from 0. For example, $|-5| = 5$, and $|3| = 3$. Determine the number of solutions of $|x||y||z| = 12$, such that x, y, and z are all integers.

Solution

Twelve can be written as a product of three positive integer factors in 18 ways, namely $1 \times 1 \times 12$, $1 \times 12 \times 1$, $12 \times 1 \times 1$, $1 \times 2 \times 6$, $2 \times 1 \times 6$, $1 \times 6 \times 2$, $2 \times 6 \times 1$, $6 \times 1 \times 2$, $6 \times 2 \times 1$, $1 \times 3 \times 4$, $3 \times 1 \times 4$, $1 \times 4 \times 3$, $3 \times 4 \times 1$, $4 \times 1 \times 3$, $4 \times 3 \times 1$, $2 \times 2 \times 3$, $2 \times 3 \times 2$, and $3 \times 2 \times 2$.

With the absolute value, each factor can by either positive or negative, so there are $2^3 \times 18 = 144$ possibilities.

5. The Main Street Math Symposium is a club that has more than one committee. Suppose that

- each committee consists of 4 members from the club.
- every pair of club members serves on exactly one committee together, and
- each pair of committees has at least one member in common.

a) Show that every two committees have exactly one member in common.

b) Show that each person is on at least 4 committees.

c) Show that each person is on at most 4 committees.

d) How many members of the club are there?

Solution

a) We are given that each pair of committees has at least one member in common. Suppose a pair of committees has two members in common. Then those two members serve on (at least) two committees together. This contradicts the given fact that every pair of members serves on exactly one committee together. Therefore every pair of committees have at most (and, therefore, exactly) one member in common.

b) Pick a committee at random and name the members A, B, C, and D. There are more than one committee, and the second committee cannot consist of the same 4 members, so it must have a new member, who we'll call E. We know E will need to be on a committee with each of A, B, C, and D, or at least 4 committees.

c) Suppose E is on a fifth committee. They are already serving on 4 committees, one with each of A, B, C, and D. Therefore, the fifth committee would not be able to contain A, B, C, or D. This is a contradiction, because the new committee would have no member in common with our original committee.

d) Let M = the number of members of the club Let C = the number of committees Then the number of committee spots total can be expressed as 4M or 4C, so M=C.

The number of pairs of committees (or members) is C(C-1)/2, which we could think of alternately as 6 pairs of members in each committee, times C committees. Therefore $C(C-1)/2 = 6C$ which can solve to get $C = M = 13$

Alternate solution for d) A, B, C, and D are on one committee all together, and exactly three other committees each, completely separately. The number of committees is $1 + 4(3) = 13$

We can also find committees that work:

ABCD AEFG AHJL AIKM

BEHI BFJM BGKM

CEJK CFLI CGHM

DELM DFHK DGJI

Note there are 13 committees and 13 distinct members.