1. The number of perfect cubes from 1 through 1,000,000 that are multiples of 7 is:

(A) 12 (B) 13 (**C**∗) 14 (D) 15 (E) 16

Solution

We note that $\sqrt[3]{1,000,000} = 100$ and that if n is a perfect cube and a multiple of 7, then $\sqrt[3]{n}$ must be a multiple of 7 as well. Multiples of 7 between 1 and 100 are $7, 14, 21, \ldots, 98 (= 7 \times 14)$, so there are 14 perfect cubes between 1 and $1,000,000$.

Answer: C

2. There are three possible Math electives to take. Alice, Bob and Carol each randomly select an elective. The probability that they all choose different electives is:

(A) $\frac{1}{9}$ (**B**∗) $\frac{2}{9}$ (C) $\frac{3}{9}$ (D) $\frac{4}{9}$ (E) $\frac{6}{9}$

Solution

The total number of combinations of 3 electives is $3 \times 3 \times 3 = 27$. The number of ways that three students take three different courses is $3 \times 2 \times 1 = 6$, because the first student has 3 choices, the second has 2 choices and the third has 1 choice. The probability is then $\dfrac{6}{27} = \dfrac{2}{9}$.

Answer: B

3. In a certain city, a taxi charges 0.20$ per 1/5 km traveled when moving faster than x km/h. It charges 0.15$ per minute when moving slower than x km/h. At x km/h, both methods of charging produce the same cost to the rider. The value of x is:

(**A**∗) 9 (B) 10 (C) 12 (D) 15 (E) 18

Solution

When the taxi is moving at exactly x km/h, the cost per hour under the first method is given by

$$\frac{\$0.20}{\frac{1}{5}\text{ km}} \cdot \frac{x\text{ km}}{\text{h}} = \$x/\text{h}.$$

Meanwhile, the cost per hour under the second method is given by

$$\frac{\$0.15}{1\text{ min}} \cdot \frac{60\text{ min}}{1\text{ h}} = \$9/\text{h}.$$

So we see that $x = 9$.

Answer: A

4. Given that $y^2 + 6x^2 = 5xy$, the largest value of $\dfrac{y+5x}{6x+y}$ is:

(A) 0 (B) $\frac{5}{6}$ (C) $\frac{7}{8}$ (**D**∗) $\frac{8}{9}$ (E) $\frac{6}{5}$

Solution

$$\begin{aligned} y^2+6x^2=5xy &\iff 6x^2-5xy+y^2=0 \\ &\iff (y-2x)(y-3x)=0 \\ &\iff y=2x \text{ or } y=3x \\ &\iff \frac{y+5x}{6x+y}=\frac{2x+5x}{6x+2x}=\frac{7}{8} \text{ or } \frac{y+5x}{6x+y}=\frac{3x+5x}{6x+3x}=\frac{8}{9} \end{aligned}$$

The largest of these is $\frac{8}{9}$ so the answer is (D).

Answer: D

5. Consider the sequence $1, 2, 2, 3, 3, 3, 4, 4, 4, 4, \ldots$, where each integer n appears n times. The 2022nd term in this sequence is:

(A) 60 (B) 61 (**C**∗) 62 (D) 63 (E) 64

Solution

The sequence has one 1, two 2's, three 3's, four 4's, etc. The following sum for example $1+2+3+4=10$ shows that there are 10 terms of 4 and below. We know that

$$1+2+3+\cdots+n=\frac{n(n+1)}{2}$$

We can check that

$$\begin{aligned} n=60 &\Rightarrow 1+2+3+...+60=\frac{60\times 61}{2}=1830 \Rightarrow 1830 \text{ terms} \le 60 \\ n=61 &\Rightarrow 1+2+3+...+61=\frac{61\times 62}{2}=1891 \Rightarrow 1891 \text{ terms} \le 61 \\ n=62 &\Rightarrow 1+2+3+...+62=\frac{62\times 63}{2}=2053 \Rightarrow 2053 \text{ terms} \le 62 \end{aligned}$$

So the terms between 1892nd term and 2053rd term are 62 which implies that 2022nd term is 62.

Answer: C

6. If the following equations are true

$A+B=1$

$B+C=2$

$C+D=3$

$D+E=4$

$E+F=5$

$\vdots$

$Y+Z=25$

then $A+Z$ equals:

(A) 14 (**B**∗) 13 (C) 12 (D) 11 (E) 10

Solution

We have

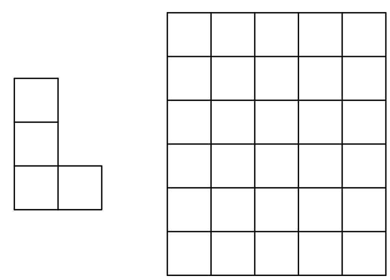
$$A + B = 1$$
$$A - C = -1$$
$$A + D = 2$$
$$A - E = -2$$
$$A + F = 3$$
$$A - G = -3$$
$$A + H = 4$$
$$\vdots$$
$$A + Z = 13$$

then $A + Z = 13$

The correct answer is (B).

Answer: B

7. The cutout shown is used to cover exactly four of the squares on the 5×6 checkerboard shown on the right. If rotations of the cutout are allowed, but not reflections, then the number of different choices for the four squares covered is:

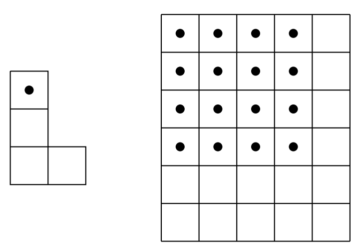


(A) 56 (B) 58 (C) 60

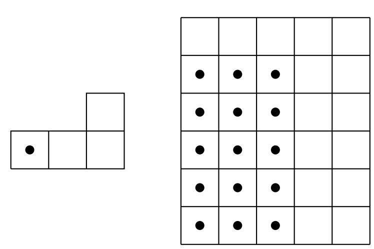
(**D**$*$) 62 (E) 64

Solution

We see that there are $4 \cdot 4 = 16$ ways to place the cutout without rotating:



If we rotate the cutout 90° counterclockwise, then there are $3 \cdot 5 = 15$ ways to place the cutout:



Rotating twice more, we get the same two pictures above but rotated 180°. So the total number of different choices is

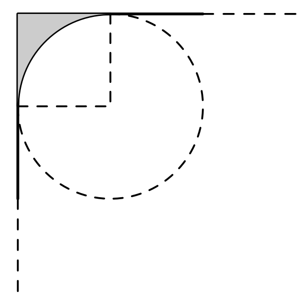
$$2(16) + 2(15) = 62.$$

Answer: D

8. A circular robotic vacuum cleaner with diameter 0.5 m moves around a room in the shape of a square 6 m on a side. The fraction of the floor area of the room that it can clean is:

(A) $\frac{576-\pi}{576}$ (**B**∗) $\frac{572+\pi}{576}$ (C) $\frac{140+\pi}{144}$ (D) $\frac{144-\pi}{144}$ (E) $\frac{32+\pi}{36}$

Solution



The only part of the floor that the vacuum cleaner cannot get at are the corners. One such corner is shown with the part that cannot be cleaned shaded. Since the diameter of the vacuum cleaner is 0.5 m, the radius is $r = 0.25$ m. Then the area not cleaned in a single corner is

$$\begin{aligned}\text{area not cleaned in one corner} &= \frac{1}{4} \times \frac{1}{4} - \frac{1}{4} \times \pi \times \left(\frac{1}{4}\right)^2 \\ &= \frac{1}{16} - \frac{\pi}{64} = \frac{4-\pi}{64}\end{aligned}$$

There are four corners, so the total area not cleaned is

$$\text{area not cleaned} = \frac{4-\pi}{16}$$

and the total area that can be cleaned is

$$\text{total area cleaned} = 6 \times 6 - \frac{4-\pi}{16} = \frac{572+\pi}{16}$$

The total area of the room is $6^2 = 36$. So the fraction that can be cleaned is

$$\text{fraction cleaned} = \frac{\left(\frac{572+\pi}{16}\right)}{36} = \frac{572+\pi}{576}$$

Note that the value of this fraction is 0.9985.

Answer: B

9. An number has the "monotonic digits" property if its digits are distinct and either increase from left to right or decrease from left to right. The number of integers between 100, 000 and one million with the "monotonic digits" property is:

(A) 84 (B) 168 (C) 192 (**D**∗) 294 (E) 306

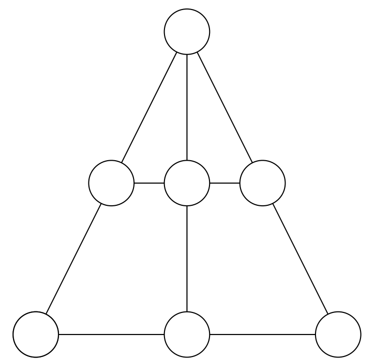
Solution

Each integer between 100, 000 and 1, 000, 000 whose digits increase from left to right has six unique digits from $\{1, 2, \ldots, 9\}$ (since it cannot start with 0), and each set of six unique digits from $\{1, 2, \ldots, 9\}$ determines exactly one such number. Each integer between 100, 000 and 1, 000, 000 whose digits decrease from left to right has six unique digits from $\{0, 1, 2, \ldots, 9\}$, and each set of six unique digits from $\{0, 1, 2, \ldots, 9\}$ determines exactly one such number. So the number of integers between 100,000 and 1,000,000 satisfying the "monotonic digits" property is the number of ways to choose six unique digits from $\{1, 2, \ldots, 9\}$, plus the number of ways to choose six unique digits from $\{0, 1, 2, \ldots, 9\}$. Therefore, the answer is

$$\binom{9}{6} + \binom{10}{6} = \frac{9!}{6! \cdot 3!} + \frac{10!}{6! \cdot 4!} = \cdot \frac{9 \cdot 8 \cdot 7}{3 \cdot 2 \cdot 1} + \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1} = 294.$$

Answer: D

10. The accompanying diagram contains several sets of circles that ”line up” (3 circles to a line). There are 5 such ”lines”. The integers from 1 through 7 are to be inserted, one number to a circle, so that the sum of the three numbers in each line is the same (this can be done in many ways). The number that can **not** be placed in the lower left circle is:



(A) 1 (B) 2 (C) 3

(**D**$*$) 4 (E) 5

Solution

Suppose that the numbers 1 through 7 have been placed in the circles so that the sum of the three numbers on each line is the same, and call this sum y. Let x be the number placed in the circle at the top. Note that the circle at the top is contained in three lines, while every other circle is contained in only two lines. Thus, summing the numbers in all five lines in two ways, we have

$$5y = 2(1 + 2 + 3 + 4 + 5 + 6 + 7) + x \quad \Rightarrow \quad 5y = 56 + x.$$

So $56 + x$ must be a multiple of 5, hence we have $x = 4$. Since we started with an arbitrary placement of the numbers satisfying the required property, we conclude that 4 must *always* be placed at the top in any such placement, hence it can *not* be placed in the lower left circle.

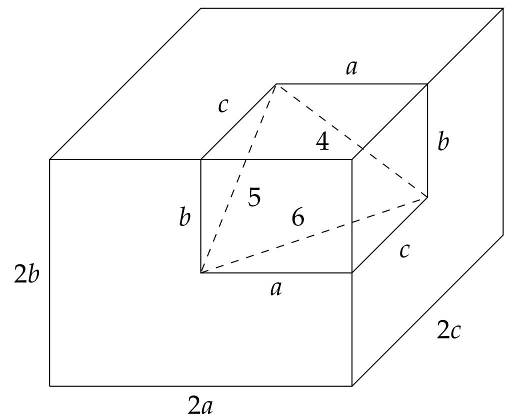
Answer: D

11. Three faces of a rectangular box meet at one corner of the box. The centres of those faces are the vertices of a triangle having sides of lengths 4, 5 and 6 cm. The volume of the box (in cm^3) is:

(A) $45\sqrt{3}$ (B) $45\sqrt{6}$ (**C**$*$) $90\sqrt{6}$ (D) 125 (E) $120\sqrt{2}$

Solution

We let the dimensions of the box be $2a$, $2b$, and $2c$. For each side of the triangle, make a right triangle with the line segments connecting to the midpoint of the nearest edge of the box, as in the diagram below.



Then by the pythagorean theorem, we have

$$\begin{cases} a^2 + b^2 = 4^2 \\ b^2 + c^2 = 5^2 \\ a^2 + c^2 = 6^2 \end{cases}$$

- Taking the first equation minus the second plus the third, we obtain $2a^2 = 27$, so $a = \sqrt{\frac{27}{2}}$.
- Taking the second equation minus the third plus the first, we obtain $2b^2 = 5$, so $b = \sqrt{\frac{5}{2}}$.
- Taking the third equation minus the first plus the second, we obtain $2c^2 = 45$, so $c = \sqrt{\frac{45}{2}}$.

Putting all of this together, the volume of the box is

$$2a \cdot 2b \cdot 2c = 2\sqrt{\frac{27}{2}} \cdot 2\sqrt{\frac{5}{2}} \cdot 2\sqrt{\frac{45}{2}} = \sqrt{2^3 \cdot 3^5 \cdot 5^2} = 2 \cdot 3^2 \cdot 5\sqrt{2 \cdot 3} = 90\sqrt{6}.$$

Answer: C

12. A function f is said to be “green" if and only if the following properties hold:

(i) $f(x)$ is defined if and only if x is an integer.

(ii) $f(x) + f(y) = f(x+y) - xy$ for all integers x and y;

(iii) $f(1) = c$ where c is some positive integer;

(iv) $f(n) = 2020$ for some integer $n \geq 2$.

The number of possible “green" functions is:

(A) 0 (B) 1 (C) 2 **(D∗)** 3 (E) 4

Solution

Substituting $x = 1$ and $y = 1$ into (ii) and using (i), we obtain

$$f(1) + f(1) = f(2) - 1 \cdot 1 \quad \Rightarrow \quad f(2) = 2c + 1.$$

Now substituting $x = 2$ and $y = 1$, we obtain

$$f(2) + f(1) = f(3) - 2 \cdot 1 \quad \Rightarrow \quad f(3) = 3c + 3.$$

Continuing in this manner, we find $f(4) = 4c + 6$, $f(5) = 5c + 10$, and so on. We notice the pattern:

$$f(n) = nc + \frac{n(n-1)}{2}.$$

We need $f(n) = 2020$ for some $n \geq 2$, so we set

$$\begin{aligned} nc + \frac{n(n-1)}{2} = 2020 \quad &\Rightarrow \quad 2nc + n^2 - n = 4040 \\ &\Rightarrow \quad n(n - 1 + 2c) = 4040. \end{aligned}$$

By definition, we need n and c to be positive integers, so n and $n - 1 + 2c$ must be factors of 4040, and we must have $n < n - 1 + 2c$. Further, we see that n and $n - 1 + 2c$ have different parity – if n is even, then $n - 1 + 2c$ is odd, and vice versa. The factors of 4040 are

$$1, 2, 4, 5, 8, 10, 20, 40, 101, 202, 404, 505, 808, 1010, 2020, 4040,$$

and since $n \geq 2$ and $n < n - 1 + 2c$, we need only consider the following pairs of factors:

- $n = 2$ and $n - 1 + 2c = 2020$: does not work, since both are even
- $n = 4$ and $n - 1 + 2c = 1010$: does not work, since both are even
- $n = 5$ and $n - 1 + 2c = 808$: gives $c = 402$
- $n = 8$ and $n - 1 + 2c = 505$: gives $c = 249$
- $n = 10$ and $n - 1 + 2c = 404$: does not work, since both are even
- $n = 20$ and $n - 1 + 2c = 202$: does not work, since both are even
- $n = 40$ and $n - 1 + 2c = 101$: gives $c = 31$

Since there are three possible values of c, there are three "green" functions.

Answer: D