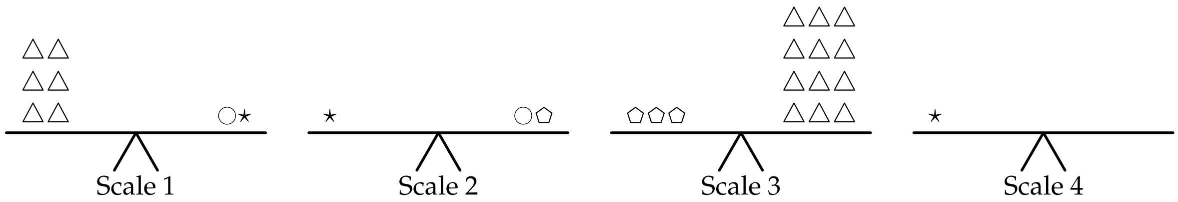
1. Scales 1, 2, and 3 are perfectly balanced. The number of triangles it will take to balance Scale 4, if the triangles are all placed on the right-hand side of the scale, is:



(A) 4 (**B**∗) 5 (C) 6 (D) 7 (E) 8

Solution

Let T = weight of a triangle, C = weight of a circle, S = weight of a star and P = weight of a pentagon.

From scale 1: $6T = C + S$,
From scale 2: $S = C + P$,
From scale 3: $3P = 12T$ which gives $P = 4T$
so scale 2 becomes $S = C + 4T$ making scale 1: $6T = 2C + 4T$ or $2T = 2C$ or $T = C$.

Now scale 2 which was originally $S = C + P$ can be written as $S = T + 4T = 5T$.

Answer: B

2. The number of real values of x for which

$$\frac{x}{x+2} = 1 + \frac{x}{2}$$

is:

(**A**∗) 0 (B) 1 (C) 2 (D) 4 (E) infinite

Solution

The equation can be written as $\dfrac{x}{x+2} = \dfrac{2+x}{x}$ which gives

$$2x = x^2 + 2x + 2x + 4 \Rightarrow x^2 + 2x + 4 = 0 \Rightarrow x = \frac{-2 \pm \sqrt{4-16}}{2}$$

This equation has no solution in real numbers because its discriminant is negative.

Answer: A

3. The usual coloring pattern on an 8×8 checkerboard is changed so that 20 unit squares are now colored red, and the rest are colored white. When the board is folded in half along a line parallel to one edge of the board, exactly seven pairs of red unit squares coincide. The number of pairs of white unit squares that coincide is:

(A) 25 (**B**∗) 19 (C) 12 (D) 7 (E) 18

Solution

We know 7 pairs of red squares coincide, so 14 red squares match and 6 squares (3 pairs) are left unmatched, so they must be mixed with white squares. Out of 64 squares, 44 of them are white which are 22 pairs but 3 of the white pairs won't match because they're mixed with three pairs of red squares. The remaining white pairs that coincide are $22 - 3 = 19$ pairs.

Answer: B

4. Simone plays a game starting with 5 blue marbles, 6 green marbles, and 7 red marbles. On each turn, Simone removes two marbles of different colors and adds one marble of the third color (e.g. she can remove one green and one blue and add one red.) She repeats this until only one marble is left. The final marble:

(A) must be blue (**B**∗) must be green (C) must be red

(D) can be blue or green, but not red (E) can be any color

Solution

Parity refers to whether a number is even or odd. We note that Red and Blue marbles have initially the same parity (both odd), and they will keep the same parity regardless of what move Simone makes. So to have 1 marble of one color and 0 of the others, the parity of colors must be different which implies that the color of the last marble must be green.

Answer: B

5. Consider the sum

$$3 + 33 + 333 + \cdots + \underbrace{333333333333333}_{15 \text{ digits}}.$$

The middle digit of this sum is:

(A) 4 (B) 5 (C) 6 (**D**∗) 7 (E) 8

Solution

First we note that the middle digit will be in the 8th column. If we write the sum as follows

$$\begin{array}{cccccccccccccccc}
 & & & & & & & & & & & & & & & 3 \\
+ & & & & & & & & & & & & & & 3 & 3 \\
+ & & & & & & & & & & & & & 3 & 3 & 3 \\
+ & & & & & & & & & & & & 3 & 3 & 3 & 3 \\
 & & & & & & & & & & & & & & & \vdots \\
+ & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3
\end{array}$$

On the first column, there are 15 3's, so they add up to 45 and 4 is carried to the column of tens.

On the second column, there are 14 3's whose sum is (14×3) plus 4 that is carried in gives 46, so another 4 is carried to the column of 100's.

On the third column, the sum is $(13 \times 3) + 4 = 43$, another 4 is carried to the fourth column.

We can summarize this in the table below

column	# of 3's	sum	number carried out
4	12	$36+4=40$	4
5	11	$33+4=37$	3
6	10	$30+3=33$	3
7	9	$27+3=30$	3
8	8	$24+3=27$	2

So the digit on the 8th column is 7.

Answer: D

6. Eight sweaters are in a box. Each sweater has one of the numbers from 1 through 8 on it, with no duplicates. If each of 4 boys grabs a sweater out of the box, the sum of the numbers of the 4 sweaters they choose could be as low as 10 or as high as 26. The probability that the sum will be at most 13 is:

(A) 0.05 (**B**∗) 0.1 (C) 0.15 (D) 0.2 (E) 0.25

Solution

The total possible combination of sweaters among 4 boys is $\dfrac{8\cdot 7\cdot 6\cdot 5}{4\cdot 3\cdot 2\cdot 1}=70$. There are 7 combinations whose sum is at most 13 (i.e. 10, 11, 12 or 13). They are as follows:

$(1,2,3,4)$ or $(1,2,3,5)$ or $(1,2,3,6)$ or $(1,2,3,7)$ or $(1,2,4,5)$ or $(1,3,4,5)$ or $(1,2,4,6)$.

So the probability is $\dfrac{7}{70}=0.1$.

Answer: B

7. Let S be the sum of all integers x such that $|4x^2-12x-27|$ is a prime number. The value of S is:

(A) -3 (B) 0 (C) 3 (**D**∗) 6 (E) 9

Solution

We have $A=4x^2-12x-27=4(x^2-3x-\frac{27}{4})$. The quadratic $x^2-3x-\frac{27}{4}$ has roots $\frac{9}{2}$ and $-\frac{3}{2}$, which gives

$$A=4\left(x-\frac{9}{2}\right)\left(x+\frac{3}{2}\right)$$

or

$$A=(2x-9)(2x+3).$$

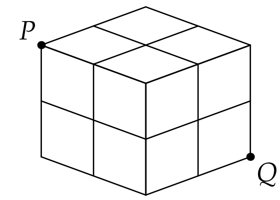
Then

$$|A|=|2x-9||2x+3|.$$

Since $|2x-9||2x+3|$ is a prime, one of its factors must be 1. Hence either $|2x-9|=1$ or $|2x+3|=1$. The first equation gives $x=5$ and $x=4$ and the second gives $x=-1$ and $x=-2$. It is easy to see that $|A|$ is prime for each of these 4 values of x. Therefore, $S=5+4-2-1=6$, so that the correct answer is (D).

Answer: D

8. Each face of a solid cube is divided into four squares, as indicated in the diagram. Starting from vertex P, paths can be travelled to vertex Q along connected line segments. If each movement along the path must move closer to Q, the number of possible paths from P to Q is:

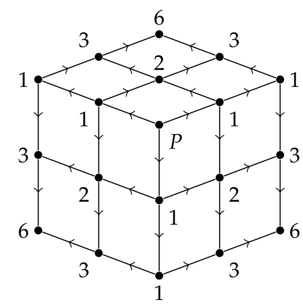


(A) 36 (B) 48 (**C***) 54

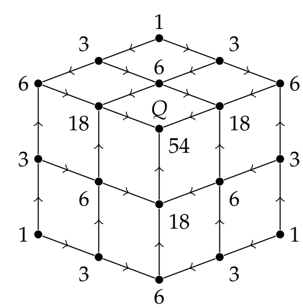
(D) 60 (E) 90

Solution

First we consider the three surfaces that touch the vertex P and count the number of paths from P to each node.



Then we turn the cube around to view the cube such that Q touches the other three surfaces, We count the number of paths to each node toward Q, which will add up to 54.



Answer: C

9. The sequence $\frac{1}{2}, \frac{5}{3}, \frac{11}{8}, \frac{27}{19}, \cdots$ is formed as follows: Each denominator is the sum of the previous numerator and denominator, and each numerator is the sum of its denominator and the denominator of the previous number. The sequence converges to the real number:

(A) $\sqrt{3}$ (B) 2 (C) $\frac{7}{5}$ (D) $\frac{3}{2}$ (**E***) $\sqrt{2}$

Solution

Given how each term of the sequence is constructed, we know that if $\frac{a}{b}$ is a term then $\frac{a+2b}{a+b}$ is the next term. If the sequence converges to some real value, then two consecutive terms must be very close, hence

$$\frac{a}{b} = \frac{a+2b}{a+b} \Rightarrow a^2 + ab = ab + 2b^2$$

so we have $a^2 = 2b^2$ or $\dfrac{a}{b} = \sqrt{2}$.

Answer: E

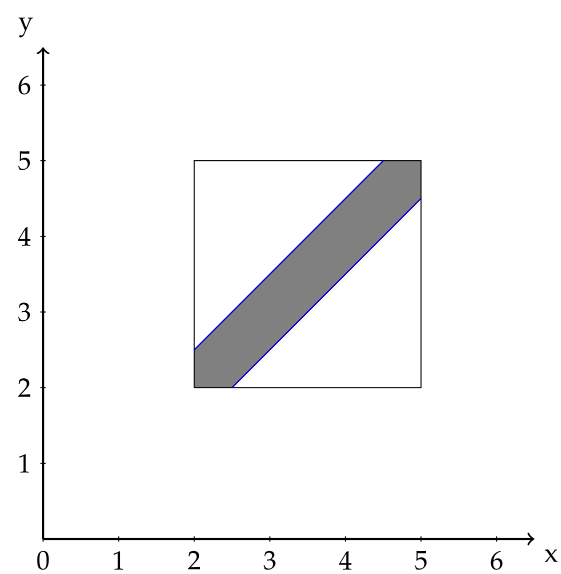
10. Two forgetful friends have agreed to meet at a coffee shop one afternoon, but each has forgotten the agreed time of the meeting and they have no way to contact one another. Each remembers the agreed time of the meeting was between 2pm and 5pm. Each of them decides to go to the coffee shop at a random time between 2pm and 5pm, wait half an hour, then leave if the other has not arrived. The probability that they meet is:

(**A**∗) $\frac{11}{36}$ (B) $\frac{1}{3}$ (C) $\frac{1}{6}$ (D) $\frac{8}{36}$ (E) $\frac{10}{36}$

Solution

let $x =$ the time that friend 1 arrives, and $y =$ the time that friend 2 arrives, then the two friends will meet if $|x - y| < \frac{1}{2}$.

Graph the lines $x - y = \frac{1}{2}$ and $x - y = -\frac{1}{2}$, the region in between is the graph of $|x - y| < \frac{1}{2}$ given below.



The area of the square is 9 and the area of the shaded region is the area of the square minus sum of the areas of the two triangles which is $9 - (2.5)^2 = 2.75$. Hence the probability that they meet is $\dfrac{2.75}{9} = \dfrac{11}{36}$.

Answer: A

==