BRITISH COLUMBIA SECONDARY SCHOOL MATHEMATICS CONTEST, 2017

Junior Final, Part A Problems & Solutions

1. A bag contains red, blue and white marbles. Exactly $\frac{3}{4}$ of the marbles are not red. Exactly $\frac{1}{3}$ of the marbles are not blue. What fraction of the marbles are not white?

   (A) $\frac{1}{6}$ (B) $\frac{5}{12}$ (C) $\frac{1}{2}$ (D) $\frac{7}{12}$ (E) $\frac{11}{12}$

   Solution

   If $\frac{3}{4}$ are *not* red, then $\frac{1}{4}$ *are* red.

   If $\frac{1}{3}$ are *not* blue, then $\frac{2}{3}$ *are* blue.

   So $\frac{1}{4} + \frac{2}{3} = \frac{11}{12}$ are not white (i.e. they *are* red or blue).

   The correct answer is (E).

   Answer: E

2. Recall that $n! = n \cdot (n-1)(n-2) \cdots 2 \cdot 1$. For example, $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$. Find a 3-digit number ABC that equals the sum $A! + B! + C!$.

   (A) 125 (B) 135 (C) 145 (D) 257 (E) none of these

   Solution

   We check, in turn, to see which one of the five choices satisfies the given equation. For the first two choices, we have

   $$1! + 2! + 5! = 1 + 2 + 120 = 123 \neq 125$$

   and

   $$1! + 3! + 5! = 1 + 6 + 120 = 126 \neq 135$$

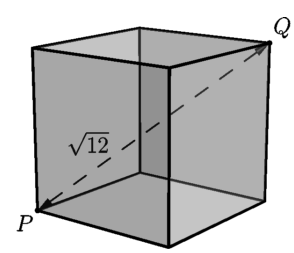
   but on the third choice we find

   $$1! + 4! + 5! = 1 + 24 + 120 = 145$$

   The correct answer is (C).

   Answer: C

3. A cube has diagonal PQ with length $\sqrt{12}$ as shown. Determine the volume of the cube.

   (A) 8 (B) 12 (C) $12\sqrt{2}$

   (D) 27 (E) $12\sqrt{2}$

   Solution

   Let x be the edge length of the cube. Applying Pythagoras' theorem to the right-triangle on the base of the cube, we find the diagonal of the base has length $L = \sqrt{x^2 + x^2} = x\sqrt{2}$. Now, by another application of Pythagoras theorem, this time to the right-triangle having PQ as its hypotenuse, we find $x^2 + (x\sqrt{2})^2 = 3x^2 = 12$, so $x = 2$, and the volume of the cube is $V = x^3 = 2^3 = 8$.

   The correct answer is (A).

   Answer: A

4. Water from a full 1.5 L bottle is poured into an empty cup until both the cup and the bottle are 3/4 full. How many litres of water are poured into the cup?

(A) $\frac{1}{4}$ (B) $\frac{3}{8}$ (C) $\frac{1}{2}$ (D) $1\frac{1}{8}$ (E) none of these

Solution

The amount of water poured into the cup equals the amount poured from the bottle. This is equal to the amount in the bottle before, minus the amount in the bottle after:

$$\frac{3}{2} - \frac{3}{4} \cdot \frac{3}{2} = \frac{3}{8} \text{ litre.}$$

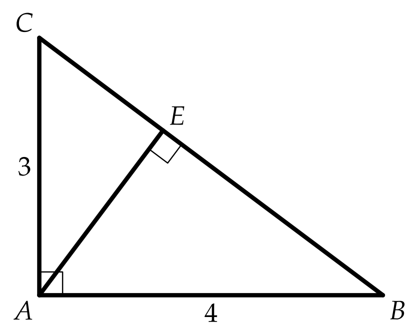
The correct answer is (B).

Answer: B

5. In a right triangle, the two shorter sides have lengths 3 and 4, and are both altitudes. How long is the third altitude of the triangle?

(A) $\frac{\sqrt{5}}{\sqrt{12}}$ (B) $\frac{5}{12}$ (C) $\frac{\sqrt{12}}{\sqrt{5}}$ (D) $\frac{12}{5}$ (E) $\frac{\sqrt{12}}{5}$

Solution



By labeling the triangle in an appropriate way, we can write $AC = 3$, $AB = 4$, and $AC \perp AB$. Since ABC is a right triangle, Pythagoras's theorem implies $BC = \sqrt{3^2 + 4^2} = 5$. Now drop a perpendicular from A to the point E on side BC. By an angle-side-angle argument, the triangles ABC and ABE are similar. It follows $\frac{AE}{4} = \frac{3}{5}$, so then $AE = \frac{12}{5}$.

Alternate solution: use Pythagoras' Theorem to find that the hypotenuse has length 5. Then we have

$$\text{are of triangle} = \frac{3 \cdot 4}{2} = 6 = \frac{5 \times \text{missing altitude}}{2}$$

so

$$\text{missing altitude} = \frac{2 \cdot 6}{5} = \frac{12}{5}.$$

The correct answer is (D).

Answer: D

6. A $3 \times 4 \times 5$ rectangular prism is painted red, then cut into sixty $1 \times 1 \times 1$ cubes. How many cubes have exactly one painted face?

(A) 22 (B) 26 (C) 38 (D) 46 (E) 47

Solution

If we remove all of the cubes around the edge of the prism, we are left with six rectangles: two of size 1 by 2; two of size 1 by 3; and two of size 2 by 3. These rectangles constitute all cubes with exactly one painted face, so there are $2 \cdot (2+3+6) = 22$ such cubes.

The correct answer is (A).

Answer: A

7. Twelve people sit around a circular table; some of them are knights, the rest are knaves. A knight always tells the truth; a knave always lies. When asked, "Are you a knight?" everyone at the table answers, "Yes." When asked, "Is the person to your right a knight?" all answer, "No." How many knights are there?

(A) 2 (B) 3 (C) 4 (D) 6 (E) 12

Solution

If there are more knights than knaves, then there must be two knights sitting next to each other in which case the knight on the left will answer "Yes" to the second question. This is a contradiction, so there cannot be more knights than knaves. Similarly, if there are more knaves than knights, then two knaves must be sitting next to each other and the knave on the left will answer "Yes" to the second question, another contradiction. It follows there must be six knights and six knaves.

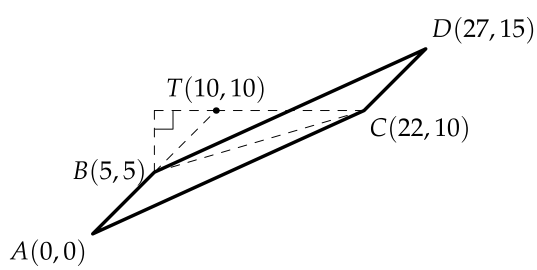
The correct answer is (D).

Answer: D

8. Let the points $(0,0)$, $(5,5)$, $(p,10)$, and (q,r) be the four vertices of a parallelogram, in some order. If the area of the parallelogram is 60, determine the largest possible value of q.

(A) -2 (B) 3 (C) 7 (D) 17 (E) 27

Solution



Let $A = (0,0)$, $B = (5,5)$, $C = (p,10)$, and $D = (q,r)$, where the four points of the parallelogram are arranged in some order. We know that ABC is half the parallelogram, and so this triangle must have area 30.

To calculate the area of ABC, construct point $T = (10,10)$. Notice that $AB = BT$, and so triangle ABC must be the same area as triangle CBT, which is equal to $(10-p) \times 5/2$ if $p < 10$ and $(p-10) \times 5/2$ if $p > 10$. As this area is 30, we have $p = -2$ or $p = 22$.

There are three ways these four points can form a parallelogram:

1. AB is parallel to CD and AC is paralled to BD
2. AC is parallel to BD and AD is paralled to BC

3. AB is parallel to CD and AD is parallel to BC

Since our goal is to maximize q, we see that the optimal solution must have $C = (22, 10)$ and $D = (27, 15)$. The answer is $q = 27$.

Answer: E

9. For how many integers n is $\dfrac{2n^2 - 13n + 20}{n^2 - 5n + 4}$ an integer?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution

By factoring and long division we have

$$\frac{2n^2 - 13n + 20}{n^2 - 5n + 4} = 2 + \frac{-3(n-4)}{(n-1)(n-4)}.$$

This is an integer if and only if $n \neq 4$ and $n - 1$ is a factor of -3. Thus $n - 1$ can take the values $-3, -1, 1, 3$. Excluding the case $n = 4$ (for which the given expression is undefined) we find that n can take the values $-2, 0, 2$.

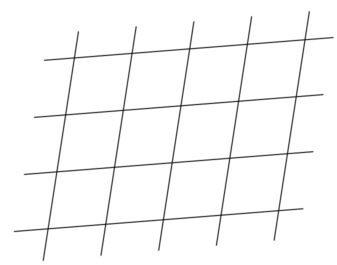
The correct answer is (C).

Answer: C

10. Five parallel lines are drawn, and then four other parallel lines are drawn in a different direction. How many distinct parallelograms are there in the picture?

(A) 30 (B) 45 (C) 52

(D) 60 (E) 100



Solution

There are $\binom{5}{2} = 10$ ways to choose two parallel sides of a parallelogram from the first direction and $\binom{4}{2} = 6$ ways to choose two parallel sides from the other direction. Hence the number of parallelograms is $10(6) = 60$; the correct answer is (D). (If the students do not know the combinations formula, they can still count like: For the first direction, there are 5 ways to choose the first of two parallel sides and 4 ways to choose the second, which gives a total count of 20, but this has to be divided by two, because the order of the two parallel sides does not matter.)

Alternative solution:

We proceed through a systematic count as follows. Call a parallelogram type (m, n) if it contains m rows and n columns of the basic (that is, smallest) type of parallelogram. The types and number of each type are

$(1,1) : 12$	$(1,2) : 9$	$(1,3) : 6$	$(1,4) : 3$
$(2,1) : 8$	$(2,2) : 6$	$(2,3) : 4$	$(2,4) : 2$
$(3,1) : 4$	$(3,2) : 3$	$(3,3) : 2$	$(3,4) : 1$

It follows the total number of parallelograms in the figure is $12 + 8 + 4 + 9 + 6 + 3 + 6 + 4 + 2 + 3 + 2 + 1 = 60$.

Answer: D