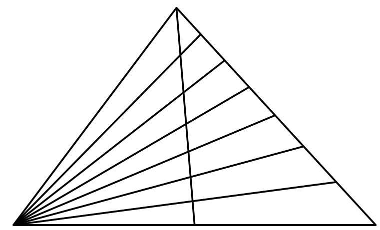
BRITISH COLUMBIA SECONDARY SCHOOL MATHEMATICS CONTEST, 2016

Senior Final, Part B Problems & Solutions

1. How many triangles appear in the diagram?



Solution

First count just triangles on the left side of the figure: there are 7 single skinny triangles, 6 "doubles," 5 "triples," 4 "four-thick," etc. for a total of $7+6+5+4+3+2+1=28$ triangles. Each of these can be extended to a triangle the whole width of the figure, which gives 28 more. On just the right side of the figure there are 7 triangles. The total is $28+28+7=63$.

Answer: 63

2. Kelly has a pear orchard. Over the last several years Kelly has found that each tree produces the same number of pears per year as there are pear trees in the orchard. Kelly divides the pears produced during each year among her seven daughters, and she keeps any pears that are left over for herself. Show that Kelly can never end up with 3 pears.

Solution

If there are T pear trees in the orchard, then T^2 pears are produced each year. We can express T as $T = 7n + r$ where n a non-negative integer and r is an integer for which $0 \leq r \lneq 7$. Then

$$T^2 = (7n+r)^2 = 49n^2 + 14nr + r^2$$

has remainder r^2 when divided by 7. We make a table of possible remainders when r^2 is divided by 7:

r	r^2	Kelly's pairs = remainder of $r^2/7$
0	0	0
1	1	1
2	4	4
3	9	2
4	16	2
5	25	4
6	36	1

Thus Kelly can only get 0, 1, 2, or 4 pears, and, hence, can never get 3, 5, or 6 pears.

Answer: See proof above.

3. Show that $n^5 - 5n^3 + 4n$ is divisible by 40 for all positive integers n.

Solution

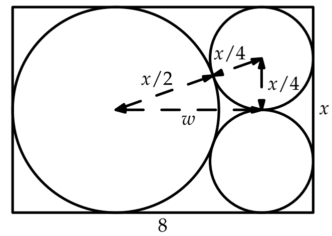
Factoring the given expression gives

$$\begin{aligned} n^5 - 5n^3 + 4n &= n(n^4 - 5n^2 + 4) \\ &= n(n^2-1)(n^2-4) \\ &= (n-2)(n-1)n(n-1)(n+1). \end{aligned}$$

Since n is an integer, the factors are five consecutive integers. Therefore one of them must be divisible by 5. Another of them is divisible by 4, and a another is divisible by 2. Since $5 \times 4 \times 2 = 40$, so the whole thing must be divisible by 40.

Answer: See proof above.

4. A rectangle has width 8 and height x. Three circles inscribed in the rectangle are tangent to each other and to the rectangle, as shown. Determine x.



Solution

From the dashed right triangle we have

$$w^2 + (x/4)^2 = (3x/4)^2 \implies w = x/\sqrt{2}$$

hence

$$8 = w + \frac{x}{2} + \frac{x}{4} = \frac{x}{\sqrt{2}} + \frac{3x}{4} \implies x = \frac{8}{1/\sqrt{2} + 3/4} = \frac{32}{2\sqrt{2}+3}.$$

Answer: $\dfrac{32}{2\sqrt{2}+3}$

5. Find all triplets (p, q, r) where p, q, and r are positive integers of which at least two are prime, for which

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r}.$$

Solution

We multiply both sides of the equation by pqr to obtain $qr + pr = pq$. There are three cases.

Case 1: p and q are prime.

We have $r(p+q) = pq$. Then, by unique factorization, $r \in \{1, p, q, pq\}$. If $r = 1$, then our equation reduces to $p + q = pq$; the only solution to this equation is $p = q = 2$. Thus, one triplet is $(2, 2, 1)$. If $r = p$ (or q), then the equation reduces to $p + q = q$ (or $p + q = p$). This implies that $p = 0$ (or $q = 0$), which contradicts the assumption that p (q) is a positive integer. Finally, if $r = pq$, then $p + q = 1$, which has no solutions over the positive integers.

Case 2: p and r are prime.

In this case, we have $pr = q(p - r)$. By unique factorization, $q \in \{1, p, r, pr\}$. If $q = 1$, then $pr = p - r$ or $r = \frac{p}{p+1}$, which has no solutions when p and r are both prime. If $q = p$, then the equation reduces to $p = 2r$, which contradicts the assumption that p is prime. If $q = r$, then we obtain $r = 0$ and another contradiction. Finally, if $q = pr$, then $p - r = 1$ or $p = r + 1$. Since the only consecutive positive integers that are both prime are 2 and 3, we have $p = 3$, $r = 2$, and the triple $(3, 6, 2)$.

Case 2: q and r are prime.

This case is analogous to Case 2 and yields the triplet $(6, 3, 2)$.

Thus, the possible triplets are $(2, 2, 1)$, $(3, 6, 2)$, and $(6, 3, 2)$.

Answer: See proof above.