BRITISH COLUMBIA COLLEGES
HIGH SCHOOL MATHEMATICS CONTEST
2002 SOLUTIONS

Junior Preliminary

1. The first and last digits must be 2. The central digits must be equal, and each could be anything from 0 to 9. Thus there are 10 palindromic numbers between 2001 and 3000.
   Answer is (**b**)

2. If I play 30 additional games, I will have played 105 games in total. To win 60% of 105 games I must win 63 games in total. Since I have already won 50 games, I need an additional 13 wins.
   Answer is (**b**)

3. Let x be the cost of the computer which generated a 10% profit, and let y be the cost of the computer which generated a 10% loss. Then

$$\begin{aligned} x + 0.1x &= 198 \qquad \text{and} \qquad & y - 0.1y &= 198 \\ 1.1x &= 198 \qquad \text{and} \qquad & 0.9y &= 198 \\ x &= 180 \qquad \text{and} \qquad & y &= 220 \end{aligned}$$

   Thus the total cost of the two computers is \$400 and the total revenue from the sales is $2 \times \$198 = \$396$. Therefore, Mark lost $\$400 - \$396 = \$4$.
   Answer is (**b**)

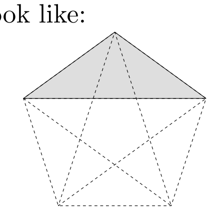
4. To be a multiple of 9, the sum of the decimal digits of the number needs to be a multiple of 9. Summing the digits we get:

$$\begin{aligned} 1+2+3+3+1+2+4 &= 16 \\ 4+6+2+3+7+4+7 &= 33 \\ 3+7+4+3+8+9+7+4 &= 45 \\ 6+7+3+4+6+4+3+8 &= 41 \\ 5+9+5+5+0+0+6 &= 30 \end{aligned}$$

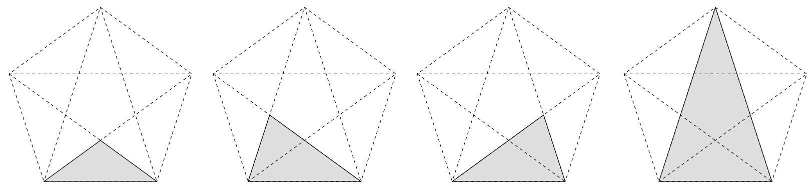
   Since 45 is the only multiple of 9 among these sums, we conclude that 37438974 is the only multiple of 9 in the list.
   Answer is (**c**)

5. If either one is lying while the other tells the truth, then either both are boys or both are girls, which is impossible. Since at least one of them is lying, we conclude that both of them are lying. Thus the one with black hair is a girl and the one with red hair is a boy.
   Answer is (**a**)

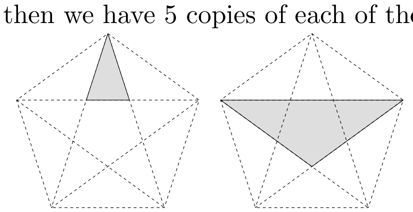
6. Let us consider 3 types of triangles initially. Those that include 2 sides of the pentagon, those that include only 1 side of the pentagon, and those that include no sides of the pentagon. If a triangle includes 2 sides of the pentagon, then those two sides must be adjacent in the pentagon and the triangle must look like:



There are 5 such triangles. If the triangle includes only 1 side of the pentagon, then for a fixed side of the pentagon we have 4 such triangles, namely:



If we consider all 5 sides of the pentagon, we get 20 such triangles. If the triangle includes no sides of the pentagon, then we have 5 copies of each of the following:



Thus there are 35 different triangles in the diagram. Answer is (**e**)

7. We first note that

$$\frac{c}{b} = \frac{9}{8} \qquad \text{and} \qquad \frac{d}{c} = \frac{3}{2}.$$

Then we have

$$\frac{ad}{b^2} = \frac{a}{b}\cdot\frac{d}{b} = \frac{a}{b}\cdot\frac{d}{c}\cdot\frac{c}{b} = \frac{3}{4}\cdot\frac{9}{8}\cdot\frac{3}{2} = \frac{81}{64}.$$

Answer is (**b**)

8. Consider the following table:

Correct	Incorrect	Unanswered	Total Score
15	0	0	75
14	0	1	71
14	1	0	70
13	0	2	67
13	1	1	66
13	2	0	65
12	0	3	63
12	1	2	62
12	2	1	61
12	3	0	60
11	0	4	59

This table considers all possible combinations which include at least 12 correct responses, together with the largest possible score that can be obtained with only 11 correct responses. We can now observe that if we replace a correct response by an incorrect response, the total drops by 5 points. Since the table shows five total scores which are consecutive numbers, we can obtain all possible totals below 59. Thus the only total scores which are not obtainable are 64, 68, 69, 72, 73, and 74. Answer is (**c**)

9. Let x be the number of children old Mr. Jones has. Then he has $3x$ grandchildren and $9x$ great-grandchildren. Since the x children each have a spouse, as do the $3x$ grandchildren, there are $4x$ spouses present. Therefore,

$$\begin{aligned} 1 + x + 3x + 9x + 4x &= 86 \\ 17x &= 85 \\ x &= 5 \end{aligned}$$

Thus there are $4x = 20$ spouses present. Answer is (**e**)

10. The area of the figure is $(x+h)^2 - h^2 = x^2 + 2xh = 60$. Since $3 < x < 5$, we have

$$9 + 6h < 60 < 25 + 10h.$$

Therefore, $h < \frac{51}{6} = 8.5$ and $h > \frac{35}{10} = 3.5$. Answer is (**d**)

11. The areas of $\triangle DBE$ and $\triangle GFC$ are each one sixth the area of $\triangle ABC$, since their altitudes are half the altitude of $\triangle ABC$ and their bases are one third of the base of $\triangle ABC$. Therefore, the area of the polygon $ADEFG$ is the area of $\triangle ABC$ less the areas of $\triangle DBE$ and $\triangle GFC$; that is, the area of $ADEFG$ is $84 - \frac{1}{6} \cdot 84 - \frac{1}{6} \cdot 84 = 56$. Answer is (**d**)

12. Let a be the length of the side of each of the two squares. Let us assume that the thickness is 1 unit. Then the one large circular cookie has volume

$$\pi \left(\frac{a}{2}\right)^2 = \frac{\pi a^2}{4}.$$

If we subdivide the other square into 16 identical squares, each will have side length $a/4$. The circular cookies made from each such small square will have volume

$$\pi \left(\frac{a/4}{2}\right)^2 = \frac{\pi a^2}{64}.$$

Since there are 16 such small cookies, we get a total volume of $\frac{\pi a^2}{4}$. Thus the ratio we seek is $1 : 1$. Answer is (**e**)

13. Let n be the number of days Samantha spent on her vacation. Her average daily expense was \$$420/n$. If she stayed for 5 more days her average daily expense would be \$$420/(n+5)$. Then we have

$$\begin{aligned} \frac{420}{n} - 7 &= \frac{420}{n+5} \\ 420(n+5) - 7n(n+5) &= 420n \\ 420n + 2100 - 7n^2 - 35n &= 420n \\ 7n^2 + 35n - 2100 &= 0 \\ n^2 + 5n - 300 &= 0 \\ (n-15)(n+20) &= 0 \end{aligned}$$

Therefore, $n = 15$ or $n = -20$. Since $n = -20$ makes no sense, we conclude that $n = 15$. Answer is (**a**)

14. The area of the base (and also of the top) of the block is 25 cm^2. Each of the four sides has area $5x$ cm^2. Hence,

$$\begin{aligned} 4(5x) + 2(25) &= 120 \\ 20x &= 70 \\ x &= 3.5 \end{aligned}$$

Answer is **(c)**

15. Let x be such a number. Then

$$\begin{aligned} \frac{1}{x} &= \frac{1}{3} \cdot x \\ x^2 &= 3 \\ x &= \pm\sqrt{3}, \end{aligned}$$

whence there are two such real numbers. Answer is **(e)**

Senior Preliminary

1. Since $AB \parallel DE$, we know that $\triangle CDE$ is similar to $\triangle CAB$. Since $DE : AB = 1 : 3$, we know that the altitudes of the triangles have the same ratio. Thus the ratio of the areas of the triangles is $1 : 9$, which means that the area of $\triangle CAB$ is $9(20) = 180$, from which we see that the area of the trapezoid $DEBA$ is $180 - 20 = 160$. Answer is **(b)**

2. The expansion of the binomial $(a+b)^6$ is given by:

$$(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6.$$

If $a = x^2$ and $b = x^{-2}$, then the term $20a^3b^3$ will contain no x, and its value will be 20. Answer is **(d)**

3. Since 3 is prime, we know that m must be a power of 3. Therefore, let $m = 3^k$. Then $m^n = (3^k)^n = 3^{20}$ implies that k divides evenly into 20. Thus, we must investigate the divisors of 20:

k	$m = 3^k$	n
1	3	20
2	9	10
4	81	5
5	243	4
10	3^{10}	2
20	3^{20}	1

Thus, we have 6 pairs (m, n). Answer is **(c)**

4. Let the y-intercept of one such 3-line be a. Then the x-intercept is $3 - a$. The slope is then

$$m = \frac{a - 0}{0 - (3 - a)} = \frac{a}{a - 3}.$$

We may also compute the slope as:

$$m = \frac{-4 - a}{-2 - 0} = \frac{a + 4}{2}.$$

Therefore,

$$\begin{aligned}\frac{a}{a-3} &= \frac{a+4}{2}\\ 2a &= a^2+a-12\\ 0 &= a^2-a-12\\ 0 &= (a-4)(a+3)\end{aligned}$$

The two possible values for a are 4 and -3, whose sum is 1. <u>Answer is (**c**)</u>

5. Let x be the width of the calendar. Then the area of the calendar is hx. Therefore,

$$\begin{aligned} ab+cd &= \tfrac{1}{2}hx\\ \text{or} \qquad x &= \frac{2ab+2cd}{h}\end{aligned}$$

<u>Answer is (**a**)</u>

6. The first painter paints $\frac{1}{2}$ m^2 in 1 minute and the second painter paints 2 m^2 in 1 minute. Together they can paint 2.5 m^2 in 1 minute. Thus, to paint 40 m^2 it takes $40/2.5 = 16$ minutes. <u>Answer is (**c**)</u>

7. Let us first list the sets of three consecutive odd primes which are less than 3000. We get

$$\begin{aligned} 3\times5\times7 &= 105\\ 5\times7\times11 &= 385\\ 7\times11\times13 &= 1001\\ 11\times13\times17 &= 2431\end{aligned}$$

We now need to count the number of multiples of these products which lie between 2001 and 3000. For the last 2 products above, there is only one multiple of each in the desired range, namely 2002 and 2431. For 385 we get only 2310 and 2695 in the desired range. For 105, on the other hand, we get nine multiples: 2100, 2205, 2310, 2415, 2520, 2625, 2730, 2835, and 2940. This gives a total of 13 multiples, except that the number 2310 has been accounted for twice in this list. Thus, there are only 12 such multiples. <u>Answer is (**e**)</u>

8. There are approximately 100 layers of tape on the roll (1 cm divided by 0.1 mm). The average length of one "layer" is $2\pi(1.5\text{ cm}) = 3\pi$ cm. The total length is then approximately 300π cm or 3π m. The closest answer among the available choices is 9. <u>Answer is (**b**)</u>

9. There are $6\times6 = 36$ possible outcomes of two rolls of the die. In 6 of these the two rolls are the same. Of the remaining 30 possible outcomes, one half (or 15) will have the first roll less than the second. Our probability is then $\dfrac{15}{36} = \dfrac{5}{12}$. <u>Answer is (**a**)</u>

10. Let the slopes of L_1 and L_2 be m_1 and m_2, respectively. Then

$$\begin{aligned} m_1 &= \frac{2p-p}{p-(p-6)} = \frac{p}{6}\\ m_2 &= \frac{2p-(-p)}{p-(p+6)} = \frac{3p}{-6} = -\frac{p}{2}\end{aligned}$$

Since L_1 and L_2 are perpendicular, we have $m_1 m_2 = -1$. That is,

$$\frac{p}{6}\cdot\left(-\frac{p}{2}\right) = -1$$
$$p = \pm\sqrt{12}.$$

Since $Q(p, 2p)$ lies in the first quadrant, we must have $p \geq 0$. Therefore, $p = \sqrt{12} = 2\sqrt{3}$.

Answer is (**c**)

11. Let x be the length in metres of the longer side of the table and let y be the length in metres of the shorter side. Then the cost of the table is $axy + bx^2$ for some constants a and b. The cost of the two sample tables gives us:

$$6a + 9b = 50$$
$$6a + 16b = 64$$

Subtracting yields $7b = 14$, or $b = 2$, whence $a = 16/3$. The cost of a square table of side length 2.5 m is then

$$\frac{16}{3}\cdot(2.5)^2 + 2\cdot(2.5)^2 = \frac{22}{3}(2.5)^2 = \frac{22}{3}(6.25) \approx \$45.83$$

Answer is (**b**)

12. Since it takes a chair 6 minutes to go 3 km, it takes 12 minutes (or 720 seconds) for a complete round trip. Since a chair leaves the starting point every 10 seconds, there are $720/10 = 72$ chairs in total. Since the skier will encounter every chair other than the one she is riding on, she will encounter 71 chairs in total. Answer is (**c**)

13. The area of the outer square is r^2. Thus, the area of the inner square is

$$\frac{1}{2}r^2 = \left(\frac{r}{\sqrt{2}}\right)^2,$$

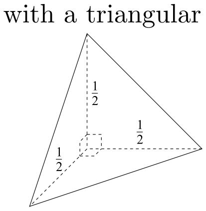
which means that its side length is $r/\sqrt{2}$. The width of the border between the 2 squares must be half the difference between these two lengths, which is

$$\frac{1}{2}\left(r - \frac{r}{\sqrt{2}}\right) = \frac{1}{2}\left(r - \frac{\sqrt{2}r}{2}\right) = \frac{1}{2}\left(\frac{2r - \sqrt{2}r}{2}\right) = \frac{r}{4}(2 - \sqrt{2}).$$

Answer is (**e**)

14. Let a, b, c be the radii of the 3 circles. Then $a + b = 5$, $b + c = 6$, and $c + a = 9$. Adding all of these together we get $2(a + b + c) = 20$, or $a + b + c = 10$. Subtracting each of the earlier equations from this, we get $c = 5$, $a = 4$, $b = 1$. The areas of the three circles are then $\pi\cdot 5^2$, $\pi\cdot 4^2$, $\pi\cdot 1^2$ for a total of 42π. Answer is (**c**)

15. Each removed piece is a pyramid with a triangular base with shape:



The area of the base is $\frac{1}{2}\cdot\left(\frac{1}{2}\cdot\frac{1}{2}\right) = \frac{1}{8}$ and the height is $\frac{1}{2}$. Its volume is then $\frac{1}{3}\cdot\left(\frac{1}{8}\cdot\frac{1}{2}\right) = \frac{1}{48}$. Since there are 8 such pyramids (one for each of the 8 corners of the original cube), their total volume is $8\cdot\frac{1}{48} = \frac{1}{6}$. Therefore, the volume of the remaining polyhedron is $\frac{5}{6}$.

Answer is (**b**)

Junior Final - Part A

1. If we write all the fractions with the same common denominator, namely 180, they become:

$$\frac{2}{5}=\frac{72}{180},\qquad \frac{5}{12}=\frac{75}{180},\qquad \frac{19}{45}=\frac{76}{180},\qquad \frac{37}{90}=\frac{74}{180}.$$

Therefore, the smallest is $\frac{2}{5}$ and the largest is $\frac{19}{45}$. The sum of these is

$$\frac{2}{5}+\frac{19}{45}=\frac{18}{45}+\frac{19}{45}=\frac{37}{45}.$$

Answer is (**c**)

2. Clearly, the number of people in his math study group is y, the number of submarine sandwiches purchased. We also know that $2x+5y=113$ and that $x>y$ and $x-y<9$. Inserting the inequalities in the equation we get:

$$\begin{aligned}113&=2x+5y>2y+5y=7y\\113&=2x+5y<2(y+9)+5y=7y+18,\end{aligned}$$

which yield $y<16\frac{1}{7}$ and $y>13\frac{4}{7}$. Thus y is one of 14, 15 or 16. But $2x$ is even and 113 is odd; therefore, $5y$ must also be odd, which means that y is odd. Hence $y=15$.
Answer is (**c**)

3. Multiplying the first equation by 2 and adding the second equation yields $11x+33=0$, whence $x=-3$. Using either equation, we then have $y=5$. Then we have

$$\begin{aligned}0&=kx-2y+4=-3k-10+4=-3k-6\\k&=-2\end{aligned}$$

Answer is (**b**)

4. Let $n=9k+7$. Then $3n-1=3(9k+7)-1=27k+20=9(3k+2)+2$. Thus, when $3n-1$ is divided by 9, we get a remainder of 2. Answer is (**b**)

5. At 2:25 the minute hand is $\frac{25}{60}$ of the way around the face of the clock, which means it is 150° from the top of the face. The hour hand is $\frac{25}{60}$ of the way between the positions 2 and 3 on the clock. Since the 2 and 3 positions on the clock are located at 60° and 90° from the top of the face, the hour hand is located at

$$60^\circ+\frac{25}{60}(90^\circ-60^\circ)=60^\circ+12.5^\circ=72.5^\circ.$$

Therefore, the angle between the two hands is $150^\circ-72.5^\circ=77.5^\circ$. Answer is (**d**)

6. Since the number of miles traveled by four tires in contact with the road while the car travels 180,000 km is $4\times 180{,}000=720{,}000$ km, and each tire can handle 120,000 km, we certainly need to use $720{,}000/120{,}000=6$ tires. The question remains as to whether we can actually change the tires in such a manner that 6 tires can do the job. This is easily seen to be possible by changing two tires every 60,000 km. Answer is (**b**)

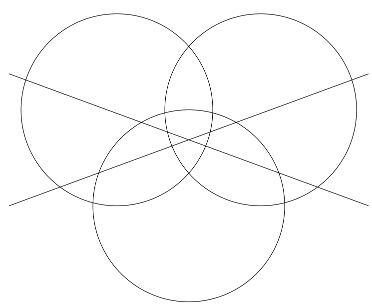
7. In order to obtain isosceles triangles we need to set the side lengths equal to each other in pairs and solve each pair for c. Setting $c = c + 3$, we clearly get no solution. Setting $c = 3c - 1$, we get $c = \frac{1}{2}$, which yields the side lengths $\frac{1}{2}$, $\frac{7}{2}$, and $\frac{1}{2}$, but these lengths do NOT actually form a triangle since one side is longer than the sum of the other two sides. (In fact, the problem states that c must be a positive integer, which also rules out $c = \frac{1}{2}$.) Lastly, we set $c + 3 = 3c - 1$ to get $c = 2$, which yields the side lengths 2, 5, 5, which does, indeed, form a triangle. Answer is (**b**)

8. Starting with the given equation, we get

$$\begin{aligned} \frac{x+y}{x-y} &= \frac{5}{8} \\ 8x + 8y &= 5x - 5y \\ 3x &= -13y \\ \frac{x}{y} &= -\frac{13}{3} \end{aligned}$$

Answer is (**a**)

9. The maximum number of points of intersection between two circles is 2. Then, with two circles intersecting in 2 points we can easily find a third circle meeting each of the first two circles in 2 points, all of which are different from each other and different from the points of intersection of the first two circles. Thus, we can construct three circles which have 6 distinct points of intersection. Now a line can meet a circle in at most 2 distinct points. Therefore, a line meeting the three circles can have at most 6 new points of intersection. The same holds for a second line. Since the two lines could have also have a point of intersection, we have a maximum of 19 points of intersection. The question remains as to whether it is possible to actually construct the three circles and two lines so as to achieve this maximum. The diagram below shows that this maximum is achievable.



Answer is (**d**)

10. Let h cm be the initial height of both candles. The first one burns at the rate of $h/4$ cm per hour, while the second one burns at the rate of $h/3$ cm per hour. If t is the number of hours since the candles were lit, then the height of the first candle is given by $h - \frac{h}{4}t$, while the height of the second candle is given by $h - \frac{h}{3}t$. We want to find the time t when the height

of the first candle is twice that of the second candle:

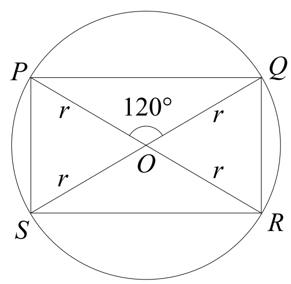
$$\begin{aligned}
h - \frac{ht}{4} &= 2\left(h - \frac{ht}{3}\right) \\
h - \frac{ht}{4} &= 2h - \frac{2ht}{3} \\
\frac{2ht}{3} - \frac{ht}{4} &= h \\
\frac{2}{3}t - \frac{1}{4}t &= 1 \\
\frac{5}{12}t &= 1 \\
t &= \frac{12}{5}
\end{aligned}$$

Since we need the time expressed in minutes we have $t = \frac{12}{5} \cdot 60 = 144$ minutes.

Answer is (**e**)

Junior Final - Part B

1. (a) Let us first draw a diagram



(b) Since QS is a diameter, we see that $\angle POS + \angle POQ = 180°$. This means that $\angle POS = 60°$.

(c) Since $OP = OS = r$, we see that $\triangle POS$ is isosceles, which means that $\angle OPS = \angle OSP$. This, coupled with the result from (b) above, means that $\triangle POS$ is equilateral, whence PS has length r.

(d) Since $\angle SPQ = 90°$, we can apply the theorem of Pythagoras to obtain

$$PQ^2 = (2r)^2 - r^2 = 3r^2,$$

which means that PQ has length $r\sqrt{3}$.

(e) The circumference of the circle is $2\pi r$. The perimeter of the rectangle is $2r + 2r\sqrt{3}$. Therefore, the ratio we seek is

$$\frac{2\pi r}{2r + 2r\sqrt{3}} = \frac{\pi}{1 + \sqrt{3}} = \frac{\pi(\sqrt{3} - 1)}{2}.$$

2. Let the four positive integers which sum to 58 be a, b, c, d. We then obtain

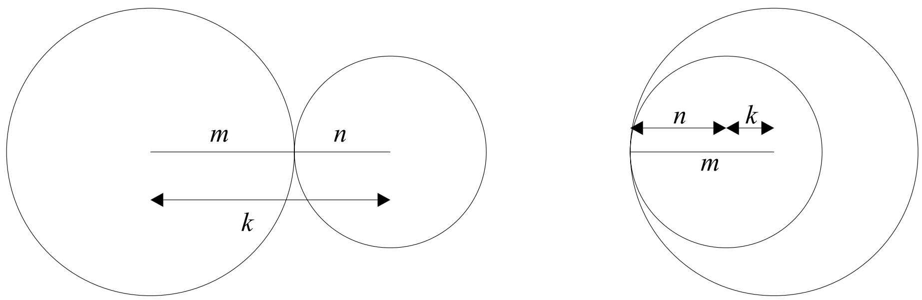
$$a + 1 = b - 2 = 3c = \frac{d}{4}.$$

Let us express each of a, b and d in terms of c using the above relationships: $a = 3c - 1$, $b = 3c + 2$, $d = 12c$. Then we have

$$\begin{aligned} a + b + c + d &= 58 \\ (3c - 1) + (3c + 2) + c + 12c &= 58 \\ 19c &= 57 \\ c &= 3, \end{aligned}$$

whence $a = 8$, $b = 11$, and $d = 36$. Therefore, the four numbers (in order) are $(a, b, c, d) = (8, 11, 3, 36)$.

3. (a) The two circles have exactly one point in common if they are tangent to each other. This can occur in precisely two ways: the point of tangency lies on the line segment joining the centres, which means that $m + n = k$; or if the centre of the smaller circle lies on the line segment joining the point of tangency to the other centre, which means that $m = n + k$. (See the diagram below.) These two conditions can be summarized by $|m - k| = n$.



(b) By a similar argument and using the diagrams above, we will get exactly two points in common if both $m + n > k$ and $m < n + k$. That is, we have two points in common precisely when $m - k > -n$ and $m - k < n$, which is equivalent to $|m - k| < n$.

4. Let a, b, and c be the numbers on the first, second, and third cans knocked down, respectively. Then the score is $a + 2b + 3c = 50$. From this, it can be seen that a and c are either both even or both odd. There are only 3 possible choices for the value of a, as one of the top row of cans must be knocked down on the first throw.

Case (i): $a = 7$.
Since c must also be odd, we cannot have $b = 9$. There remains two choices for b, namely $b = 10$ and $b = 8$. If $b = 10$, then the score for these two throws is $7 + 2(10) = 27$, which leaves 23 for the remaining throw. But the score on the third throw is a multiple of 3, so this case is ruled out. If $b = 8$, then the score for these two throws is $7 + 2(8) = 23$, which leaves a remainder of $27 = 3(9)$. Since a can numbered 9 is available (and there is only one such can available), case (i) leads to exactly one possible way to score 50.

Case (ii): $a = 10$.
Since c must also be even, we must have $c = 8$ or $c = 10$; the former can only occur if $b = 7$, while the latter can only occur if $b = 8$. This results in a score of

$$10 + 2(7) + 3(8) = 48 \qquad \text{or} \qquad 10 + 2(8) + 3(10) = 56,$$

neither of which is acceptable, since we must score 50.

Case (iii): $a = 8$.
Then $b = 7$ or $b = 10$. Since c must also be even, we must have $c = 10$. This results in a score of

$$8 + 2(7) + 3(10) = 52 \qquad \text{or} \qquad 8 + 2(10) + 3(10) = 58,$$

neither of which is acceptable.

Thus, there is only one possible ordering of the numbers to get 50, namely 7 on the first throw (only one choice) followed by 8 on the second throw (only one choice), and 9 on the third throw (again, only one choice). Therefore, there is only one possible way to get a score of 50.

Alternate Approach: Since all the cans have numbers which are at least as large as 7 and no larger than 10, we can let the numbers on the first, second, and third throws be $x + 7$, $y + 7$, and $z + 7$, where $0 \le x, y, z \le 3$. Then we have

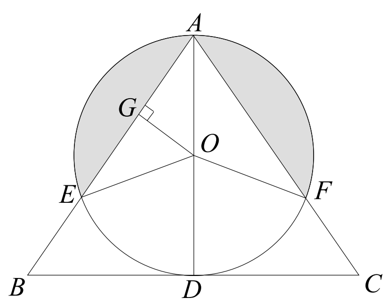
$$\begin{aligned}(x + 7) + 2(y + 7) + 3(z + 7) &= 50\\ x + 2y + 3z &= 8\end{aligned}$$

Now let us consider all possible integer solutions to this last equation with all the integers between 0 and 3, inclusive. The only solutions are easily found to be:

$$\begin{aligned}(x, y, z) &= (0, 1, 2), \quad (1, 2, 1), \quad (2, 3, 0), \quad (2, 0, 2), \quad (3, 1, 1)\\ (x + 7, y + 7, z + 7) &= (7, 8, 9), \quad (8, 9, 8), \quad (9, 10, 7), \quad (9, 7, 9), \quad (10, 8, 8).\end{aligned}$$

Since there is no 9 in the top row, we can eliminate the two with $x + 7 = 9$. Since there are only two cans numbered 8 among all the paint cans, and one of them is in the third row, the only way to get it knocked down is to knock down an entire column of cans successively. Since the other can numbered 8 is not in that column, it is impossible to score 50 points with two cans numbered 8. This leaves only one possibility, namely $(7, 8, 9)$, and there is only one way to obtain it.

5. Let AB and AC intersect the circle at E and F, respectively, as shown in the diagram below.



Let O be the centre of the circle and join O to E and F. Since BC has length 4, so do AB and AC. We may then use the Theorem of Pythagoras to determine that the length of AD is $\sqrt{4^2 - 2^2} = \sqrt{12} = 2\sqrt{3}$. Thus, the circle has radius equal to $\sqrt{3}$. Since $\triangle ABC$ is equilateral we see that $\angle BAD = 30°$. Since OA and OE are both radii of the circle we see that $\triangle OAE$ is isosceles, whence $\angle AEO = 30°$. Therefore, $\angle AOE = 120°$. Similarly, $\angle AOF = 120°$. Thus, the area of the sector AOE is one third the area of the circle, and the same is true for sector AOF; that is, both are equal to $\frac{1}{3}\pi(\sqrt{3})^2 = \pi$. On the other hand, if we drop a perpendicular from O to the line AE meeting it at G, then we see that $\triangle ABD$ is similar to $\triangle AOG$. Therefore,

$$\frac{AO}{AB} = \frac{AG}{AD}$$
$$\frac{\sqrt{3}}{4} = \frac{AG}{2\sqrt{3}}$$
$$AG = \frac{3}{2}$$

Thus, $AE = 2{\cdot}AG = 3$, and by the Theorem of Pythagoras we have

$$OG = \sqrt{\left(\sqrt{3}\right)^2 - \left(\frac{3}{2}\right)^2} = \sqrt{3 - \frac{9}{4}} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}.$$

Therefore, the area of $\triangle AOE = \frac{1}{2}{\cdot}3{\cdot}\sqrt{3}/2$. Then the shaded area adjacent to the segment AE has area

$$\pi - \frac{3\sqrt{3}}{4},$$

which means that the total area is twice that, namely

$$2\pi - \frac{3\sqrt{3}}{2}.$$

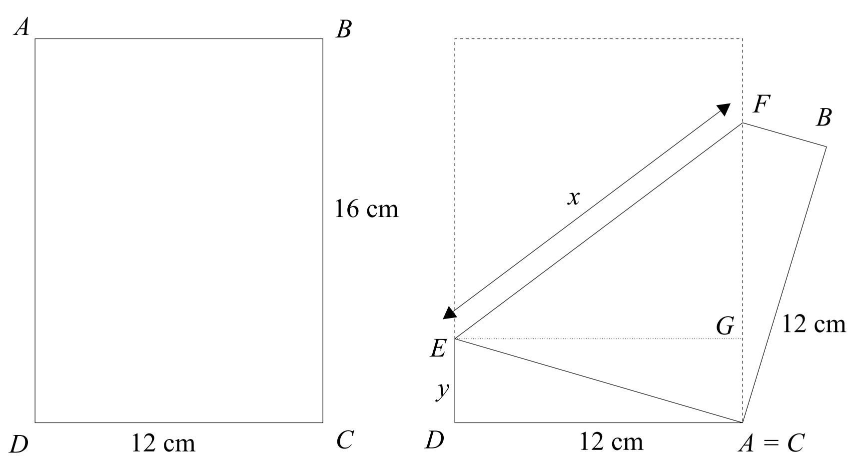
Senior Final - Part A

1. Let the two-digit number be represented by ab where a is the first digit and b is the second. Then the number is actually $10a + b$. We are told that

$$10a + b = 7(a + b)$$

This simplifies to $3a = 6b$, or $a = 2b$. Because a is a decimal digit, we know that $a < 9$, whence $b \leq 4$. On the other hand, we must have $a > 0$ in order to have a two-digit number. This means that $b \geq 1$. Therefore, we have such a number for all values of b between 1 and 4, inclusive: $ab = 21$, 42, 63, 84. Answer is **(e)**

2. Let us first draw a diagram (below) and let x be the length of the fold, as shown in the diagram.



Let E and F be the points where the fold meets the lines DA and BC, respectively, as in the diagram, and let y cm be the length of the line segment DE. Then the line segment AE has length $16-y$ cm. Applying the Theorem of Pythagoras to $\triangle CDE$ we get

$$\begin{aligned}(16-y)^2 &= y^2+12^2\\ 256-32y+y^2 &= y^2+144\\ 112 &= 32y\\ y &= 3.5\end{aligned}$$

Since $\angle DAF = \angle EAB = 90°$, we see that $\angle DAE = \angle BAF$, which means that $\triangle DAE$ is congruent to $\triangle BAF$ (since AB and CD have the same length). Therefore, FA has length $16-y$ cm and BF has length y cm. Now let G be the point where the line passing through E parallel to CD meets the line FC. Then $\triangle EFG$ is right-angled with hypotenuse of length x and legs of length 12 and $16-2y=9$. The Theorem of Pythagoras then yields:

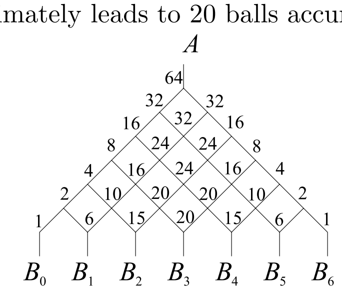
$$\begin{aligned}x^2 &= 12^2+9^2 = 144+81 = 225\\ x &= 15\end{aligned}$$

Answer is (**d**)

3. The given expression is positive when the numerator and denominator have the same sign. Let us first consider the case when the denominator, $x-4$, is positive, i.e. $x>4$. In this case $x^2>16$, which clearly means that $x^2-9>0$. Thus $x>4$ is part of the solution. Now let us consider the other case, namely $x<4$. Since the denominator is negative, we want the numerator to be negative also. That is, we want $x^2<9$. This occurs if and only if x lies strictly between -3 and 3. Answer is (**d**)

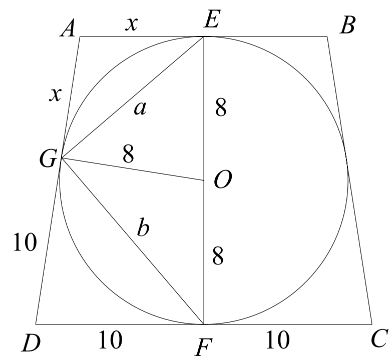
4. Since $AB \parallel DE$ and $DE:AB = 1:3$, we see that $\triangle CDE$ is similar to $\triangle CAB$, whence we also have $CD:CA = 1:3$. Then $\triangle CDE$ and $\triangle CAE$ have the same altitude from the vertex E, while their bases are in the ratio $1:3$. Thus, the area of $\triangle CAE$ is three times the area of $\triangle CDE$. That is, the area of $\triangle CAE$ is $3\times 20=60$. Since $\triangle DEA$ has area equal to the area of $\triangle CAE$ minus the area of $\triangle CDE$, we see that the area of $\triangle DEA$ is $60-20=40$. Answer is (**b**)

5. The 64 balls which enter at A will be split in half with 32 balls traveling along each of the 2 branches. When the 32 balls get to the next “fork” they will split again, this time into 16 going each way. The 16 balls which went left at the first “fork” and right at the second will meet up with the 16 balls which went right at the first “fork” and left at the second. Thus the balls will be arranged as 16, 32, 16 at the next level. This process continues as shown in the diagram below, which ultimately leads to 20 balls accumulating at B_3.



Answer is (**d**)

6. Let $ABCD$ be the isosceles trapezoid with $AB \parallel DC$ and $BC = AD$. Let O be the centre of the inscribed circle. Let EF be a diameter of the circle with E on AB and F on CD, as in the diagram below.



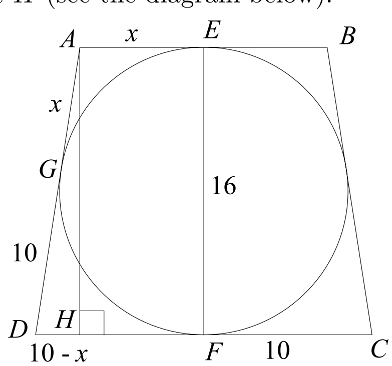
Let G be the point of tangency between the inscribed circle and the line AD. Let x be the length of the segment AE. Since the tangents from a point to a circle have the same length, we see that AG also has length x, and that DG has length 10. Let GE and GF have lengths a and b, respectively. Since $AE \perp OE$ and $OG \perp AG$, we see that $\angle GAE + \angle GOE = 180°$. Similarly, we have $\angle GDF + \angle GOF = 180°$. Since it is clear that $\angle GOE + \angle GOF = 180°$, we see that $\angle GAE = \angle GOF$ and $\angle GDF = \angle GOE$. Because all four of these triangles are isosceles triangles, we see that $\triangle GAE$ is similar to $\triangle GOF$ and that $\triangle GDF$ is similar to $\triangle GOE$. From this we conclude that

$$\frac{x}{a} = \frac{8}{b} \qquad \text{and} \qquad \frac{10}{b} = \frac{8}{a}.$$

The first of these simplifies to $x = 8a/b$, while the second yields $a/b = 4/5$. Thus, we have $x = 32/5 = 6.4$. Now the area of the trapezoid is

$$\begin{aligned} \text{Area} &= \frac{1}{2} \cdot 16 \cdot (2x + 20) = 16(x + 10) \\ &= 16\left(\frac{32}{5} + 10\right) = 16\left(\frac{82}{10}\right) = \frac{1312}{5} \end{aligned}$$

Alternate Approach: We again define E, F, G, and x as above. Next we drop a perpendicular from A to DC meeting DC at H (see the diagram below).



Clearly the length of the segment DH is $10 - x$. Applying the Theorem of Pythagoras to $\triangle ADH$ we obtain

$$\begin{aligned}(10+x)^2 &= 16^2 + (10-x)^2\\ 100 + 20x + x^2 &= 256 + 100 - 20x + x^2\\ 40x &= 256\\ x &= \frac{32}{5}\end{aligned}$$

The rest of the solution continues as above. Answer is **(a)**

7. Observe first that there are only 6 possibilities for the choices of the types of faces of the polyhedron, given as ordered pairs of (triangles, squares): $(0,5)$, $(1,4)$, $(2,3)$, $(3,2)$, $(4,1)$, and $(5,0)$, since the total number of faces is 5. Now each edge of the polyhedron belongs to exactly two faces. Therefore, if we sum the number of edges for each face, we count each edge twice, whence we have exactly twice the number of edges of the polyhedron. In particular, the sum of the number of edges for each face must be even. Hence, there must be an even number of triangular faces, since each contributes a value of 3 (which is odd) to this sum, while squares contribute a value of 4 (which is even). This eliminates three of the six possibilities, namely $(1,4)$, $(3,2)$, and $(5,0)$. Further, if the polyhedron has only square faces, each vertex of the polyhedron is a vertex belonging to exactly three faces. Thus, if we sum of the number of vertices for each face, each vertex will be counted three times, and we must get a multiple of three. Since there are exactly five square faces, the sum of the number of vertices for each face is 20, which is NOT a multiple of 3. This eliminates $(0,5)$. That leaves two possibilities, namely $(2,3)$ and $(4,1)$.
Let us now start with any square face. Then this square plus the four faces adjacent to it account for all the faces of the polyhedron. If we are dealing with the case $(4,1)$, the remaining four faces are all equilateral triangles, and we have a pyramid with a square base. Now, let us consider the case $(2,3)$. Two of the faces adjacent to our initial square must be triangular and two must be square. In fact, the same is true for each of the three squares. This means that each of the two triangles is adjacent to all three squares. The resulting solid is sometimes called a "triangular prism". In any event, there are exactly two possible polyhedra with five faces all of whose faces are either squares or equilateral triangles.
Answer is **(c)**

8. Since CD is tangent to the arc BC at C, we know that $\angle ACD = 90°$. Since $\triangle ABC$ is equilateral, we know that $\angle ACB = 60°$. Therefore, $\angle BCD = 30°$. The area we want is clearly the sum of the areas of the equilateral triangle $\triangle ABC$ and the circular sector BCD minus the circular sector BAC. Since the area of the sector BAC is one-sixth of the area of the full circle and the area of the sector BCD is one-twelfth of the area of the full circle, and since both circular sectors have radius 2 units, we see that the area of the sector BAC is $\frac{1}{6}\cdot\pi\cdot 2^2 = \frac{2}{3}\pi$, whence the area of the sector BCD is $\frac{1}{3}\pi$. It remains to find the area of $\triangle ABC$. By constructing an altitude to one side and using the Theorem of Pythagoras, we compute the altitude of $\triangle ABC$ to be $\sqrt{3}$. Thus, the area of $\triangle ABC$ is $\frac{1}{2}\cdot 2\cdot\sqrt{3} = \sqrt{3}$. Hence, the area of the shaded region is $\sqrt{3} + \frac{1}{3}\pi - \frac{2}{3}\pi = \sqrt{3} - \frac{1}{3}\pi$. Answer is **(a)**

9. Since the skier is descending more rapidly than the chair lift, she will encounter every chair on the lift exactly once on her way down. Therefore, there are 72 chairs in total on the lift. Assuming that the chairs are equally spaced on the lift, this means that there are 36 chairs ascending and 36 chairs descending the lift at any given time. Thus, it takes $36 \times 10 = 360$ seconds, or 6 minutes, for a chair to ascend or descend the lift. This means that the chair

is moving at the rate of 0.5 km/min, or 30 km/h. On her descent the skier will definitely encounter the 36 chairs that are on their way up the lift, no matter what her speed is. But she encounters an additional 15 chairs. Therefore, it has taken her $15 \times 10 = 150$ seconds, or 2.5 minutes, to descend the 3 km. Her average speed then is $3/2.5 = 1.2$ km/min, or 72 km/h.

Alternate Approach: As above we note that the chair is moving at the speed of 30 km/h. Let v be the speed in km/h of the skier going down the hill. We are told that $v > 30$. Let t_o be the time for the skier to travel between successive chairs moving in the opposite direction to the skier. The distance traveled by the skier from one such chair to the next is clearly vt_o. However, we also know that this distance traveled by the skier plus the distance traveled by the chair during this time must be the same as the distance between successive chairs on the lift, i.e. the distance traveled by any chair in 10 seconds. Therefore,

$$vt_o + 30t_o = 30 \cdot 10,$$

which means that $t_o = 300/(v + 30)$. If we let t_s be the time for the skier to travel between successive chairs moving in the same direction as the skier, a similar series of calculations will yield $t_s = 300/(v - 30)$. The total time taken for her descent is then $51t_o = 21t_s$. Therefore,

$$\begin{aligned} \frac{51 \cdot 300}{v + 30} &= \frac{21 \cdot 300}{v - 30} \\ 17(v - 30) &= 7(v + 30) \\ 10v &= 720, \end{aligned}$$

whence the skier travels at $v = 72$ km/h. Answer is (**d**)

10. Since the dogs either always tell the truth or lie alternately, and the answer to the same question elicits two different responses from Ariel, we conclude that Ariel is the alternating liar today. In addition, we know that the other two dogs will tell the truth. Unfortunately, we cannot determine from the given information whether Ariel started with a lie or with the truth. Consequently, (a) and (e) can NOT be concluded. Since asking Winnie, who tells the truth, whether Ariel ate the homework could yield either a YES or NO response, and a NO response means there are still two possible culprits, we can NOT conclude that (b) or (d) are correct. We may, however, conclude that Ariel ate the homework with a YES response, which means that (c) is correct. Answer is (**c**)

Senior Final - Part B

1. Let each of the small rectangles have length x m and width y m, where $x > y$. Then the width of the large rectangle shown is $x + y$, and the length is $3x$ and also $4y$. Thus, $y = \frac{3}{4}x$, which makes the area of the large rectangle (in terms of x alone):

$$\text{Area} = 336 = 3x\left(x + \frac{3}{4}x\right) = \left(3 + \frac{9}{4}\right)x^2 = \frac{21}{4}x^2$$

from which we get $x^2 = 64$, whence $x = 8$ and $y = 6$. The dimensions of the large rectangle are then $x + y = 14$ and $3x = 4y = 24$, which means the perimeter is $14+14+24+24 = 76$ m.

2. If an accumulated sum of at least 16 is facing either player, then that player can obviously win by choosing the difference between that sum and 22, which is certainly an integer between

1 and 6, inclusive. This means that if the accumulated sum facing a player is 15, then that player must lose, since he must choose a number which puts the sum into the range from 16 to 21, inclusive. Therefore, if a player can get a sum of 15, he has (effectively) won the game. This means that 15 plays the same role as 22 in the game. A similar analysis shows that 8 also plays this role, as does 1. Therefore, the first player should choose 1, and thereafter choose whatever is needed to obtain the sums of 8, 15, and 22, successively.

3. Since $A'C' \parallel AC$, we see that $\triangle A'DB$ is similar to $\triangle ABC$. Let k be the common ratio:

$$k = \frac{A'D}{AC} = \frac{BD}{BC}$$

The area of $\triangle A'DB$ will be k^2 times the area of $\triangle ABC$, since both the base and the altitude of $\triangle A'DB$ have been reduced by a factor of k. Therefore, we need $k^2 = \frac{1}{2}$. That is, $k = 1/\sqrt{2}$. In particular, $A'B = AB/\sqrt{2} = 10/\sqrt{2} = 5\sqrt{2}$, whence

$$AA' = AB - A'B = 10 - 5\sqrt{2}.$$

4. By looking at the units column of the addition, we conclude that $3C = C$, $3C = C + 10$, or $3C = C + 20$, which yield the values $C = 0$, $C = 5$, or $C = 10$, respectively. Since C must be a single decimal digit, the last of these is impossible and we have $C = 0$ or $C = 5$. However, by considering the leading digits of the addition, we see that $C > P$, which means that $C \neq 0$. Therefore, $C = 5$. Knowing that $C = 5$ means that we must carry 1 into the tens column of the addition: thus, from the tens column we have $3I + 1 = 5$, $3I + 1 = 15$, or $3I + 1 = 25$. The only one of these which can be solved for an integer value is the last one, which means that $I = 8$.

Let h be the carry out of the hundreds column and let t be the carry out of the thousands column. By examining the thousands column, we have $I + L + C + h = C + 10t$, which is equivalent to $L + h = -8 + 10t$. Since L must be a single decimal digit and we know that $0 \leq h, t \leq 2$, we deduce that $t = 1$ and $L + h = 2$. Thus, $0 \leq L \leq 2$. Furthermore, the hundreds column then implies that $F + 2T + 2 = C + 10h$, or $F + 2T = 3 + 10h$, from which we deduce that F is odd.

Now consider the ten-thousands column. We have $C+A+R+1 = 15$ or $C+A+R+1 = 25$, which yield $A+R = 9$ or $A+R = 19$. The latter is impossible for the sum of two decimal digits. Therefore, we have $A + R = 9$, and we have a carry of 1 out of this column. This means that $2A+B+1 = 15$ or $2A+B+1 = 25$, which yield $2A+B = 14$ or 24, respectively, implying that B is an even digit. We also know that P is C (i.e. 5) minus the carry into the last column on the left, which is 1 or 2 depending on whether $2A+B = 14$ or 24, respectively. Let us now consider all possible values for the quadruple (A, B, P, R). (Recall that B is an even digit.) If $B = 0$, then $2A = 14$ or 24, whence $A = 7$, $R = 2$, and $P = 4$. If $B = 2$, then $2A = 12$ or 22, whence $A = 6$, $R = 3$, and $P = 4$. If $B = 4$, then $2A = 10$ or 20, whence $A = 5$, which is impossible (since $C = 5$). If $B = 6$, then $2A = 8$ or 18, whence $A = 4$ and $R = 5$, which is impossible (since $C = 5$) or $A = 9$, $R = 0$, and $P = 3$. Since $I = 8$, we need not consider $B = 8$. Therefore, the only possibilities for (A, B, P, R) are $(7, 0, 4, 2)$, $(9, 6, 3, 0)$, and $(6, 2, 4, 3)$. Let us consider each case separately.

Case (i) $(A, B, P, R) = (7, 0, 4, 2)$.
Since $0 \leq L \leq 2$, we see that L must be 1 (since $B = 0$ and $R = 2$), whence $h = 1$ and $F + 2T = 13$. Since F is odd, $F = 3$ or $F = 9$ (since $L = 1$, $C = 5$, and $A = 7$). If $F = 3$,

then $T = 5$, which is impossible (since $C = 5$). If $F = 9$, then $T = 2$, which is impossible (since $R = 2$). Thus, case (i) is impossible.

Case (ii) $(A, B, P, R) = (9, 6, 3, 0)$.

Since $0 \le L \le 2$ and $B = 0$, we see that $(L, h) = (1, 1)$ or $(2, 0)$. Thus, $F + 2T = 3$ (if $h = 0$) or 13 (if $h = 1$). Since F is odd, then $F = 1$ or $F = 7$ (since $P = 3$, $C = 4$, and $A = 9$). If $F = 1$, then $T = 1$ which is impossible (since $F = 1$), or $T = 6$ which is impossible (since $B = 6$). If $F = 7$, then $T = 3$ which is impossible (since $P = 3$). Thus, case (ii) is impossible.

Case (iii) $(A, B, P, R) = (6, 2, 4, 3)$.

Since $0 \le L \le 2$ and $B = 2$, we see that $(L, h) = (0, 2)$ or $(1, 1)$. Thus $F + 2T = 13$ (if $h = 1$) or 23 (if $h = 2$). Since F is odd, then $F = 1$, $F = 7$, or $F = 9$. If $F = 1$, then $T = 6$ which is impossible (since $A = 6$). If $F = 7$, then $T = 3$ which is impossible (since $R = 3$), or $T = 8$ which is impossible (since $I = 8$). If $F = 9$, then $T = 2$ which is impossible (since $B = 2$), or $T = 7$, $h = 2$, and $L = 0$. This gives the only possible solution:

$$4658985 + 260785 + 635785 = 5555555$$

5. (a) For 2-digit decreasing numbers, we need to begin with any of the digits from 1 through 9 and then observe that if the first digit is 1, there is only one possibility, namely 10; if the first digit is 2, we have two possibilities, namely 20 and 21; if the first digit is 3, we have three possibilities, namely 30, 31, and 32; and so on, up to nine possibilities if the first digit is 9. This gives us a total of $1 + 2 + \cdots + 9 = 45$ decreasing 2-digit numbers. We next note that when one removes the first digit of any decreasing number with 3 or more digits, the remaining number must also be a decreasing number with 1 fewer digits, and whose leading digit is smaller then that of the number we started with. Thus, for 3-digit decreasing numbers, if we start with leading digit 2 (the smallest possible), we get only one possibility, namely 210; if the leading digit is 3, then we have $1 + 2 = 3$ possibilities, namely 310, 320, and 321; if the leading digit is 4, then we have $1 + 2 + 3 = 6$ possibilities, and so on, up to $1 + 2 + \cdots + 8 = 36$ possibilities for leading digit 9. (The numbers 1, 2, 3, etc. used in the above sums are the numbers obtained for the number of 2-digit decreasing numbers beginning with the given digit.) This gives a total of

$$1 + 3 + 6 + 10 + 15 + 21 + 28 + 36 = 120$$

3-digit decreasing numbers.

(b) It is clear that since the digits must all be distinct, and there are only 10 possible decimal digits, the largest number of digits for any decreasing number is 10. Indeed, there is only one decreasing number with 10 digits, namely 9876543210.

(c) Continuing in the same vein as in part (a), we may build the following table, in which the column headings represent the number of digits in the decreasing number, and the row headings represent the leading digit of the decreasing number:

	2	3	4	5	6	7	8	9	10
1	1								
2	2	1							
3	3	3	1						
4	4	6	4	1					
5	5	10	10	5	1				
6	6	15	20	15	6	1			
7	7	21	35	35	21	7	1		
8	8	28	56	70	56	28	8	1	
9	9	36	84	126	126	84	36	9	1
Total	45	120	210	252	210	120	45	10	1

Each entry in the table beyond the first column is the sum of all those entries in the previous column which lie in the rows above it. The total number of decreasing numbers is:

$$45 + 120 + 210 + 252 + 210 + 120 + 45 + 10 + 1 = 1013.$$

Remark: The enterprising student may also note that if we seek the number of n-digit decreasing numbers, we may start by choosing n digits from the 10 available, which can be done in $\binom{10}{n}$ ways. Having made this selection, we also note that there is precisely one decreasing number composed of these digits. Thus, the number of n-digit decreasing numbers is $\binom{10}{n}$. This means the answer to (a) is $\binom{10}{2} = 10{\cdot}9/2 = 45$, and the answer to (c) is:

$$\sum_{n=2}^{10} \binom{10}{n} = \binom{10}{2} + \binom{10}{3} + \cdots + \binom{10}{10},$$

which can be easily computed and summed. Again, the enterprising student might observe that this is also

$$\sum_{n=0}^{10} \binom{10}{n} - \binom{10}{0} - \binom{10}{1} = \sum_{n=0}^{10} \binom{10}{n} - 1 - 10 = 2^{10} - 11 = 1024 - 11 = 1013,$$

since the sum of the entries in the n^{th} row of Pascal's triangle is 2^n.