BRITISH COLUMBIA COLLEGES

High School Mathematics Contest

1999 – Solutions

Preliminary Round – Junior

1. Let x be the number of correct answers and y the number incorrect. Then $x + y \leq 20$, $x \geq 0$, $y \geq 0$ and $5x - 2y = 48$. For this equation to have integer solutions, x must be even and

$$\begin{aligned} 5x &= 48 + 2y \leq 48 + 2(20 - x) = 88 - 2x \\ &\Rightarrow 7x \leq 88 \Rightarrow x \leq 12 \quad \text{since } x \text{ is an integer} \end{aligned}$$

Thus the maximum number of questions that can be answered correctly is $\boxed{12}$.

.... (C)

2. Let S be the area of the shaded region. It can be calculated by subtracting the area of the rectangle $ABCD$ from the area of the quarter circle. By the Pythagorean theorem the radius of the circle is $BD = \sqrt{CD^2 + BC^2} = \sqrt{4^2 + 3^2} = 5$. Hence

$$S = \frac{1}{4}\pi \times 5^2 - 3 \times 4 = \frac{25\pi}{4} - 12 = \frac{25\pi - 48}{4}$$

But $76 \leq 25\pi \leq 80$ so that

$$\frac{76 - 48}{4} \leq S \leq \frac{80 - 48}{4} \Rightarrow \boxed{7 \leq S \leq 8}$$

.... (D)

3. The areas of one slice of the pizzas is summarized in the table:

Size of pizza	Area of one slice
8-inch	$\frac{1}{3}\left(4^2\pi\right) = \frac{16}{3}\pi$
10-inch	$\frac{1}{4}\left(5^2\pi\right) = \frac{25}{4}\pi$
12-inch	$\frac{1}{6}\left(6^2\pi\right) = 6\pi$
14-inch	$\frac{1}{8}\left(7^2\pi\right) = \frac{49}{8}\pi$

The areas are thus proportional to $5\frac{1}{3}$, $6\frac{1}{4}$, 6 and $6\frac{1}{8}$. Obviously, the largest number is $6\frac{1}{4}$ which corresponds to the $\boxed{\text{10-inch}}$ pizza.

.... (B)

4. Suppose that you are one of the eight people, you hold the \$30, and you have decided to give the other seven as little of the money as possible, just to satisfy the imposed conditions. Obviously, in that way you will be left with the largest share. You can place yourself in one of the three categories of people:

   1. You are the person who gets more than \$5.

2. You are amoung the four people who get more than \$1.
3. You are in a group other than 1 or 2 above.

Clearly, you will need to give away the least amount of money if you are the person who gets more than \$5. In that case you need to give away \$2 to four other people, \$1 to the remaining three people, and you are left with $\$30 - 4 \times \$2 - 3 \times \$1 = \boxed{\$19}$

.... (D)

5. Arrange the fifty children in a grid as shown below, with the given data entered in bold face type and x as the number of brown-eyed brunettes.

	blonde	brunette	Totals
blue-eyed	**14**	$31 - x$	$14 + (31 - x)$
brown-eyed		x	**18**
Totals		**31**	**50**

The other cells in the table are filled in based on the given data. Thus

$$14 + (31 - x) + 18 = 50 \Rightarrow 63 - x = 50 \Rightarrow x = \boxed{13}$$

.... (D)

6. Refer to a row in the diagram by giving its endpoints, e.g., the row with endpoints 3 and 9 will be denoted $(3, 9)$ and its sum is $3 + 8 + 6 + 9 = 26$. This is the sum of all of the rows. For row $(1, 9)$ the sum is $1 + 5 + D + 9 = 26 \Rightarrow D = 11$. From row (C, E) we get $C + 6 + D + E = 26 \Rightarrow C + E = 9$. Numbers 1, 3, 4, 5, 6, 8, 9, and 11 are already placed on the diagram, hence C and E must be among the numbers 2, 7, 10 and 12. Since $C + E = 9$ there are only two possiblities: $C = 2$ and $E = 7$, or $C = 7$ and $E = 2$. The first possibility, together with the sums of rows $(4, C)$ and $(4, E)$ give $A = 12$ and $B = 10$, while the second possibility gives $A = 7$ and $B = 15$. But we must have $B \leq 12$, so the second possibility does not give a valid solution. Note that with $A = 12$ and $B = 10$, row $(1, 3)$ has a sum of 26. Thus 7 goes in the place marked $\boxed{\text{E}}$.

.... (E)

7. Since Mark is y places in front of Sam, there are $x - y$ people behind Sam, excluding Sam. Then, since there are z people in front of Sam, again excluding Sam, the total number of people in the line are:

$$\text{number behind Sam} + \text{number in front of Sam} + \text{Sam} = \boxed{x - y + z + 1}$$

.... (D)

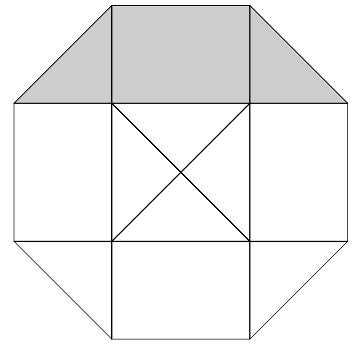
8. We can list the possibilities for the values of the angles A, B and C:

A	B	C
20	20	140
20	40	120
20	60	100
20	80	80
40	40	100
40	60	80
60	60	60

Thus there are $\boxed{7}$ possibilities.

.... (B)

9. The octagon can be decomposed into four equal rectangles and eight equal triangles, as shown on the diagram. The triangles are congruent isosceles right triangles with hypotenuses equal to the length of the side of the octagon. The rectangles are equal by the symmetry of the octagon. If R and T represent the area of one of the rectangles and one of the triangles, respectively, then the ratio of shaded area to the total area of the octagon is



$$\frac{R+2T}{4R+8T} = \boxed{\frac{1}{4}}$$

.... (B)

10. By placing an **X** anywhere else than A, a line with two **X**'s and no **O**'s is created, increasing **X**'s chances to lose. This suggests that placing an **X** at $\boxed{\text{A}}$ is the move that guarantees that **X** does not lose. In order to prove that this is indeed the case, we to verify two claims:

    1. If **X** starts at A, then he can get at least a draw no matter how **O** continues to play.
    2. If **X** does not start at A then **O** can win no matter how **X** continues to play.

    To prove the first claim, we need to consider all possible moves of **O** and only specific"defenses" by **X**. To prove the second, we need to consider all possible moves of **X** and specific "attacks" by **O**. The solution is presented in two tables:

X starts at A

X	O	X	O	X	Result
A	B	E	C	D	draw
A	B	E	D	C	draw
A	C	D	B	E	draw
A	C	D	E	B	draw
A	D	C	B	E	draw
A	D	C	E		O loses
A	E	B	C	D	draw
A	E	B	D		O loses

X does not start at A

X	O	X	O	X	X always loses. The line with three X's.
B	C	A	D	E	BXE
B	C	D	A	E	BXE
C	B	A	E	D	CXD
C	B	E	A	D	CXD
D	E	A	B	C	CXD
D	E	B	A	C	CXD
E	C	A	D	B	BXE
E	C	D	A	B	BXE

The second table does not contain the games in which **X** loses deliberately in the second move. The strategy of O can be described briefly as "forcing **X** to complete the line with two **X**'s, obtained by placing the **X** in a spot other than A."

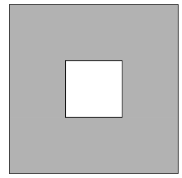
.... (A)

11. Obviously, there are 99 girls and only one boy at the institute. Since 2% of students living on campus are boys, the boy must live there, and he forms the total of the male population of the campus. Thus, the 2% of the population living on campus consists of one student. So $\frac{1}{0.2} = 50$ students live on campus and $100 - 50 = \boxed{50}$ students live off campus.

.... (D)

12. The time required for the job is directly proportional to the number of houses and inversely proportional to the number of painters. Hence, if three painters can paint four houses in five days, then one painter can paint one house in $\frac{5\times3}{4}$ days, and seven painters will need $\frac{5\times3\times18}{4\times7} = \frac{15\times9}{14} = 9\frac{9}{14}$ days to paint 18 houses. Thus to the nearest day it would take $\boxed{10}$ days for seven painters to paint 18 houses.

.... (D)

13. The surface area of the solid consists of six faces of the form  and the walls of six holes which can be unfolded into the form . Thus the total area is $6\,(3\times 3 - 1\times 1) + 6\times 4 = 48 + 24 = \boxed{72}$.
    .... (C)

14. Let d be the number of days in a week on Aardvark, w be the number of weeks in a month, and m be the number of months in a year. The given conditions translate into three equations: $d = w$, $m = 2dw$, $dwm = 1250$. Hence, $m = 2d^2$ and $d^2m = 1250$. Further, by combining the last two equations, $2d^4 = 1250$. This gives $d^4 = 625 = 5^4$ or $d = 5$, and consequently, $m = 2\left(5^2\right) = 50$.
    .... (A)

15. Observe that the units digit of $n!$ is zero for $n \geq 5$. Indeed, if $n \geq 5$, then it is a multiple of 10 since $n! = 1\times 2\times 3\times 4\times 5\times\cdots\times n$. Therefore, the units digit in the sum $1! + 2! + 3! + \cdots + 1999!$ is the same as the units digit in the sum $1! + 2! + 3! + 4! = 1 + 2 + 6 + 24 = 33$. Thus the units digit is $\boxed{3}$
    .... (B)

Preliminary Round – Senior

1. A single red plum costs $\frac{1}{4}$ of a dollar in either store, and a yellow plum costs $\frac{1}{3}$ of a dollar in the first store and $\frac{1}{6}$ of a dollar in the second store. Since you bought m red plums and n yellow plums in each store, you bought $2m + 2n$ plums for a total cost of \$10. Thus

$$\underbrace{\frac{m}{4}+\frac{n}{3}}_{\text{first store}} + \underbrace{\frac{m}{4}+\frac{n}{6}}_{\text{second store}} = 10 \Rightarrow \frac{m}{2}+\frac{n}{2} = 10 \Rightarrow m+n = 20 \Rightarrow 2m+2n = 40$$

   Thus you bought $\boxed{40}$ plums altogether.
    .... (C)

2. See Problem 10 on the Junior Preliminary test.
    .... (A)

3. Note that

$$\begin{aligned} 2^{36}-1 &= \left(2^3\right)^{12}-1 = \left[\left(2^3\right)^6-1\right]\left[\left(2^3\right)^6+1\right] = \left[\left(2^3\right)^3+1\right]\left[\left(2^3\right)^3-1\right]\left[\left(2^3\right)^6+1\right] \\ &= \left(2^3+1\right)\left[\left(2^3\right)^2-2^3+1\right]\left[\left(2^3\right)^3-1\right]\left[\left(2^3\right)^6+1\right] = 9k \end{aligned}$$

   Thus $2^{36}-1$ is divisible by nine, so that when the number $68a19476735$ is divided by nine the remainder is zero. By the principle of "casting out nines" this remainder equals the remainder when the sum of the digits in the number is divided by 9. But $6+8+a+1+9+4+7+6+7+3+5 = a+56$. Since when 56 is divided by 9 the remainder is 2, this is the same as $a+2$. For this to be divisible by nine the digit a must $\boxed{7}$.
    .... (E)

4. Writing each of the logarithm equations in exponential form gives:

$$\begin{aligned}\log_9 20 &= a \Leftrightarrow 9^a = 20 \Leftrightarrow 3^{2a} = 20\\ \log_3 n &= 4a \Leftrightarrow n = 3^{4a} = \left(3^{2a}\right)^2 = 20^2 = \boxed{400}\end{aligned}$$

.... (A)

5. Since the numbers on the blackboard are 5 and 8, the numbers on the players hats are between 0 and 8. If either player sees 6, 7 or 8 on the other person's hat, she immediately knows that the sum is 8 and thus knows her own number. This would be reported on the first bell. If the first bell brings no response, then both players know that neither has 6, 7 or 8 on her hat. After the first bell, if either player sees 0, 1 or 2 on the other's hat, she knows the sum is 5, since otherwise her number would be 8, 7 or 6 to give a sum of 8, so that she knows the number on her hat. This would be reported on the second bell. If there is no response on the second bell, then both players know that neither has 0, 1, 2, 6, 7 or 8 on her hat so that they know that the only possible numbers for either of them are 3, 4 or 5. This means that the sum is 8 and each player then knows her number. This would be reported on the third bell. Thus the bell rings at most $\boxed{3}$ times.

.... (B)

6. See Problem 7 on the Junior Preliminary test

.... (D)

7. Let x and y be the two numbers and assume $x \geq y$. Then $x - y : x + y : xy = 1 : 7 : 18$, so that if $x - y = k$ we have $x + y = 7k$ and $xy = 18k$. Adding the first two equations gives $x = 4k$ and putting this into the third equation gives $4ky = 18k$. Obviously $k \neq 0$, since otherwise both x and y will be zero, so that last equation gives $y = \frac{9}{2}$. Since $x = 4k$ we have $y = 3k$ so that $k = \frac{1}{3}y = \frac{3}{2}$ and $x = 6$. Thus $xy = \boxed{27}$. This answer does not appear among those given.

Alternate solution: Using the given ratios we have $x - y = \frac{1}{18}xy$ and $x + y = \frac{7}{18}xy$. Hence,

$$(x+y)^2 - (x-y)^2 = \left(\tfrac{7}{18}xy\right)^2 - \left(\tfrac{1}{18}xy\right)^2 \Rightarrow 4xy = \frac{4}{27}(xy)^2$$

Since $xy \neq 0$ this gives $xy = \boxed{27}$.

.... (E)

8. Since $\overline{AB} = 1$, the quarter circle ABG has area $\frac{\pi}{4}$. Similarly, the quarter circle DFC has area $\frac{\pi}{4}$. Then

$$\begin{aligned}\text{Area of } ABCD &= ABG + DFC - EFG + BEC\\ &= ABG + DFC = \boxed{\frac{\pi}{2}}\end{aligned}$$

.... (B)

9. We have $a^2 - b^2 = (a-b)(a+b) = 1999$. Since 1999 is a prime number its only divisors are 1 and 1999, so that $a - b = 1$ and $a + b = 1999$ and

$$\begin{aligned}a^2 + b^2 &= \frac{1}{2}\left[(a+b)^2 + (a-b)^2\right] = \frac{1}{2}\left(1999^2 + 1^2\right)\\ &= \frac{1}{2}\left[(2000-1)^2 + 1\right] = \frac{1}{2}(4000000 - 4000 + 2)\\ &= \frac{1}{2}(3996002) = \boxed{1998001}\end{aligned}$$

Alternate solution: $a - b = 1$ and $a + b = 1999 \Rightarrow a = 1000$ and $b = 999$, so that

$$\begin{aligned} a^2 + b^2 &= 1000^2 + 999^2 = 1000000 + (1000-1)^2 \\ &= 1000000 + 1000000 - 2000 + 1 = \boxed{1998001} \end{aligned}$$

.... (B)

10. Use the equation $s = vt$, where s is the distance, v the velocity and t the time. Let v_s be Sam's velocity relative to the water and v_r be the velocity of the river. Then

$$\frac{s}{v_s - v_r} = 6 \qquad \text{and} \qquad \frac{s}{v_s + v_r} = 3$$

and the time for Sam to swim the distance s upstream and float downstream the same distance is $6 + \dfrac{s}{v_r}$. The two equations above can be written

$$\frac{v_s}{s} - \frac{v_r}{s} = \frac{1}{6} \qquad \text{and} \qquad \frac{v_s}{s} + \frac{v_r}{s} = \frac{1}{3}$$

Subtracting the first of these equations from the second gives

$$2\frac{v_r}{s} = \frac{1}{3} - \frac{1}{6} = \frac{1}{6} \Rightarrow \frac{v_r}{s} = \frac{1}{12} \Rightarrow \frac{s}{v_r} = 12$$

Hence the time it takes Sam to swim the distance s upstream and float the same distance downstream is $6 + 12 = \boxed{18}$.

.... (D)

11. Factoring gives $x^2 - 6x + 8 = (x-2)(x-4)$. We must find the segments of the interval $-10 \le x \le 10$ where $(x-2)(x-4) \ge 0$. We divide the interval into three subintervals: $-10 \le x \le 2$, $2 < x \le 4$, $4 < x \le 10$. Consider the table below:

	$-10 \le x \le 2$	$2 < x \le 4$	$4 < x \le 10$
$x-2$	$-$	$+$	$+$
$x-4$	$-$	$-$	$+$
$(x-2)(x-4)$	$+$	$-$	$+$

Thus we see that $(x-2)(x-4) \ge 0$ for $-10 \le x \le 2$ and $4 \le x \le 10$. The probability that x is a solution to the inequality is equal to the ratio of the sum of the lengths of both intervals to the length of the interval $-10 \le x \le 10$, that is $\dfrac{12+6}{20} = \boxed{0.9}$.

.... (E)

12. Let $x + 2y = a$ and $2x + y = b$. Then $ab = 27$, where a and b are the integer divisors of 27. Thus the possible values of (a, b) are: $(1, 27)$, $(-1, -27)$, $(3, 9)$, $(-3, -9)$, $(9, 3)$, $(-9, -3)$, $(27, 1)$, $(-27, -1)$. Since the system of two equations is symmetric with respect to x and y, a solution (x, y) corresponds to a given pair (a, b) if and only if (y, x) corresponds to (b, a). Consequently, we need to find only the solutions corresponding to the first four values of (a, b), then the remaining solutions can be obtained by switching x and y. By solving the system for x and y in terms of a and b, we get

$$x = \frac{2b - a}{3} \qquad \text{and} \qquad y = \frac{2a - b}{3}$$

The pairs $(1, 27)$ and $(-1, -27)$ do not yield integer solutions. The pair $(a, b) = (3, 9)$ gives $(x, y) = (5, -1)$, and $(a, b) = (-3, -9)$ gives $(x, y) = (-5, 1)$. By symmetry, the other solutions are $(-1, 5)$ and $(1, -5)$, for a total of $\boxed{4}$ solutions.

.... (D)

13. Each interior angle of a regular pentagon is $\frac{1}{5}(5 \times 180^\circ - 360^\circ) = 108^\circ$. Since $\angle PAE = \angle PEA = 60^\circ$, we see that $\angle BAP = \angle DEP = 48^\circ$. Since $PA = PE = AE = BA = BD$, we see that $\triangle BPA$ and $\triangle PDE$ are isosceles triangles each with a vertex angle of 48°. Thus $\angle BPA = \angle PBA = \frac{1}{2}(180^\circ - 48^\circ) = 66^\circ$. Similarly, $\angle DPE = 66^\circ$. Therefore, $\angle BPD = 360^\circ - 2 \times 66^\circ - 60^\circ = \boxed{168^\circ}$.

.... (A)

14. The sum of two numbers is even if both numbers are even or both are odd. The probability that Jack's number is even is $\frac{5}{10} = \frac{1}{2}$. Given that his number is even, four of the nine remaining balls have an even number, so the probability of Jill drawing an even numbered ball is $\frac{4}{9}$. Thus, the probability that both balls have even numbers is $\left(\frac{1}{2}\right)\left(\frac{4}{9}\right) = \frac{2}{9}$. In the same way, the probability that both balls have odd numbers is also $\frac{2}{9}$. Since the events "both numbers are even" and "both numbers are odd" are disjoint, the probability that the sum is even is $\frac{2}{9} + \frac{2}{9} = \boxed{\frac{4}{9}}$.

Alternate solution: No matter which ball Jack removes, there will be four remaining out of the nine left which yield an even sum. Thus Jill has a probability of $\boxed{\frac{4}{9}}$ to remove a ball that yields an even sum.

.... (A)

15. Since $\angle DAE + \angle ADE = 90^\circ = \angle DAE + \angle BAE$, we see that $\angle ADE = \angle BAE$ and thus $\triangle ADE$ is similar to $\triangle BAF$. This means that

$$\frac{DE}{AE} = \frac{AF}{BF} \qquad \text{or} \qquad \frac{5}{3} = \frac{AF}{BF}$$

Similarly,

$$\frac{CF}{BF} = \frac{DE}{CE} = \frac{5}{7}$$

Therefore

$$\begin{aligned} \frac{AF}{BF} + \frac{CF}{BF} &= \frac{AF + CF}{BF} = \frac{AE + CE}{BF} = \frac{10}{BF} = \frac{5}{3} + \frac{5}{7} = \frac{50}{21} \\ &\Rightarrow BF = \frac{210}{50} = \frac{21}{5} = \boxed{4.2} \end{aligned}$$

.... (C)

Final Round – Junior Part A

1. When DC is diluted 1 to 4 then one fifth of the mixture is DC, so there are are twelves glasses worth of DC in 60 glasses of the mixture. If the same twelve glasses are diluted 1 to 5, the DC makes up one sixth of the mixture. Thus there are a total of $\boxed{\text{72 glasses}}$ of the mixture.

.... (C)

2. Let A be the unshaded area from the small circle and B be the unshaded area from the large circle. Then

$$A + \frac{\pi}{3} = \pi \Rightarrow A = \frac{2\pi}{3} \quad \text{and} \quad B + \frac{\pi}{3} = 9\pi \Rightarrow B = \frac{26\pi}{3}$$

so that the total area of the unshaded regions is $A + B = \boxed{\frac{28\pi}{3}}$.

.... (D)

3. The perimeter of $\triangle PQR$ is

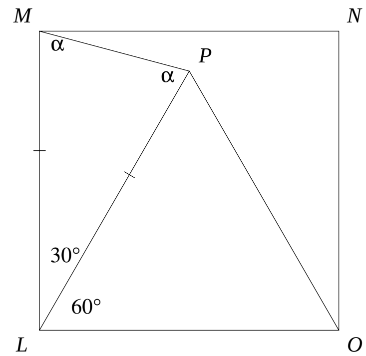
$$\begin{aligned} PQ+QR+RP &= (PM-QM)+QR+(PN-RN) = PM+PN+QR-QM-RN \\ &= PM+PN+QR-QX-XR \end{aligned}$$

But $QR = QX + XR$ so that $PQ+QR+RP = PM+PN = \boxed{27}$.

.... (D)

4. Construct the line segments LP, OP and MP. Since $\triangle LPO$ is equilateral $LP = LO = LM$. Thus $\triangle LPM$ is isosceles with $\angle MPL = \angle LMP = \alpha$. Further $\angle PLO = 60°$, so that $\angle PLM = 30°$. Thus

$$2\alpha + 30° = 180° \Rightarrow \alpha = \boxed{75°}$$



.... (A)

5. If Ali broke the toy, then Barbara is lying by saying that Tyler broke. So Ali cannot have broken the toy. If Tyler broke it, then Ali is lying. So Tyler cannot have broken it. If Hei-Lam broke it, then both Ali and Barbara are lying. So Hei-Lam cannot have broken it. If Barbara broke it, then nobody else is lying. Thus it must have been $\boxed{\text{Barbara}}$ who broke the toy.

.... (B)

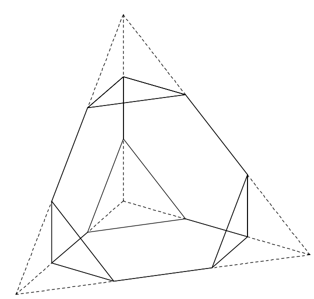
6. The number 3^{19} is a multiple of 9, so that, by the principle of "casting out nines", the sum of its digits must be a multiple of nine. Thus $1+1+a+2+2+6+1+4+6+7 = 30+a$ must be a multiple of nine where a is a single digit, i.e., a nonnegative integer less than ten. Hence $a = \boxed{6}$.

.... (D)

7. Let α be the angle between the hour hand and the horizontal and β be the angle between the minute hand and the vertical. Then, since the hour hand moves through 30° in one hour, $\alpha = 30° \left(\frac{26}{60}\right) = 13°$, and, since the hour hand moves through 360° in one hour and the minute hand is four minutes before the vertical, $\beta = 360° \left(\frac{4}{60}\right) = 24°$. Hence the acute angle between hands at 3:26 is $90° - \alpha - \beta = \boxed{53°}$. This value does not appear in the list of answers.

.... (E)

8. The original tetrahedron has six edges and four vertices. Each corner that is cut off introduces at most three additional edges. Hence there are at most $6 + 4 \times 3 = \boxed{18}$ edges.



.... (D)

9. Numbers less than 10^6 have at most six digits. Two digit palindromes are of the form aa where a is a nonzero digit, so there are 9 two digit palindromes. Three digit palindromes are of the form aba where a is as before and b is any digit, so there are $9 \times 10 = 90$ three digit palindromes. Four digit palindromes are of the form $abba$ where a and b are as before, so there are 90 four digit palindromes. Five digit palindromes are of the form $abcba$ where both b and c can be any digits, so there are $9 \times 10 \times 10 = 900$ five digit palindromes. Six digit palindromes are of the form $abccba$ where both b and c can be any digits, so there are $9 \times 10 \times 10 = 900$ six digit palindromes. Thus the total number of palindromes with six or less digits is $9 + 90 + 90 + 900 + 900 = \boxed{1989}$.

. . . . (D)

10. Using the clerk's method the reduced price is $(1 - 0.21) \times (1 + 0.7) = 0.79 \times 1.07 = 0.8453$ times the original price. Using the customer's way the reduced price is $(1 + 0.7) \times (1 - 0.21) = 0.79 \times 1.07 = 0.8453$ times the original price, which is the same as the clerk's method. The compromise gives a reduced price of $1 - 0.14 = 0.86$ times the original price. Thus the
$\boxed{\text{the compromise is the worst while the other two ways are equally good}}$.

. . . . (B)

Final Round – Junior Part B

1. In a triangle the sum of the lengths of any two sides is more than the length of the other side. Thus for an isosceles triangle the sum of the lengths of the equal sides must be less than the length of the other side. For an isosceles triangle sides of integral length the sides can be: $(19, 19, 2)$, $(18, 18, 4)$, $(17, 17, 6)$, $(16, 16, 8)$, $(15, 15, 10)$, $(14, 14, 12)$, $(13, 13, 14)$, $(12, 12, 16)$, and $(11, 11, 18)$. This gives a total of $\boxed{9}$ possible triangles.

2. If there are n pages in the book, then by the formula given the sum of the page numbers is $1+2+\cdots+n = \dfrac{n\,(n+1)}{2}$. Find the largest n so that this sum is less than 1999. Since $\sqrt{2 \times 1999} = \sqrt{3998} \approx 63$, start at 62

$$\frac{62 \times 63}{2} = 1953, \qquad \frac{63 \times 64}{2} = 2016$$

Thus there are 62 pages and the page that is counted twice is $1999 - 1953 = \boxed{46}$. Note if there were 61 pages then the page that was counted twice would have to be $1999 - 61 \times \frac{62}{2} = 1999 - 1891 = 108$ which is not a page in the book.

3. Extending the labeled sides of the equilateral triangle as shown gives another equilateral triangle whose side length is

$$10 + 19 + 4 = 10 + 8 + 15 = 15 + 14 + 4 = \boxed{33}$$

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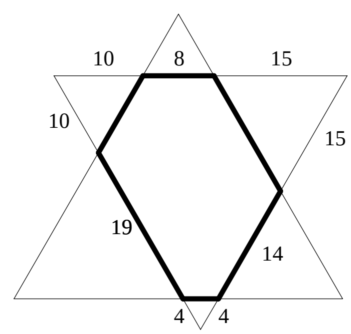


4. Let N be the number of eggs in the basket. Then $N - 1$ must be divisible by 2, 3, 4, 5 and 6. Since the lowest common multiple of these numbers is 60, $N - 1$ must be a multiple of 60. Thus we must find an N less than 500 such that $N - 1$ is a multiple of 60 and N is divisible by 7. The possibilities are: $1 \times 60 + 1 = 61$, $2 \times 60 + 1 = 121$, $3 \times 60 + 1 = 181$, $4 \times 60 + 1 = 241$, $5 \times 60 + 1 = 301$,

$6 \times 60+1 = 361$, $7 \times 60+1 = 421$, and $8 \times 60+1 = 481$. The only one of these integers that is divisible by 7 is $301 = 7 \times 43$. Thus there must be $\boxed{301}$ eggs in the basket.

Alternate solution 1: The total number of eggs must end in the digit 1 since it is one more than a multiple of 5 and a multiple of 2. Since the total is a multiple of 7 ending in a 1 that is 500 or less, it must be one of 7×3, 7×13, 7×23, 7×33, 7×43, 7×53 or 7×63. Checking, we find that only $7 \times 43 = 301$ satisfies all of the conditions.

Alternate solution 2: The number of eggs N must satisfy the following system of congruences

$$\begin{aligned} N &\equiv 0 \bmod 7 \\ N &\equiv 1 \bmod 5 \\ N &\equiv 1 \bmod 4 \\ N &\equiv 1 \bmod 3 \end{aligned}$$

The first congruence gives $N = 7y$, so that the second gives

$$7y \equiv 1 \bmod 5 \equiv 21 \bmod 5 \Rightarrow y \equiv 3 \bmod 5 \Rightarrow y = 3 + 5z$$

Then the third congruence gives

$$7(3+5z) \equiv 1 \bmod 4 \equiv 21 \bmod 4 \Rightarrow 3+5z \equiv 3 \bmod 4 \Rightarrow z \equiv 0 \bmod 4 \Rightarrow z = 4w$$

and the fourth gives

$$\begin{aligned} 7(3+20w) &\equiv 1 \bmod 3 \equiv 7 \bmod 3 \Rightarrow 3+20w \equiv 1 \bmod 3 \equiv 4 \bmod 3 \\ &\Rightarrow 20w \equiv 1 \bmod 3 \equiv 40 \bmod 3 \Rightarrow w \equiv 2 \bmod 3 \\ &\Rightarrow w = 2 + 3t \end{aligned}$$

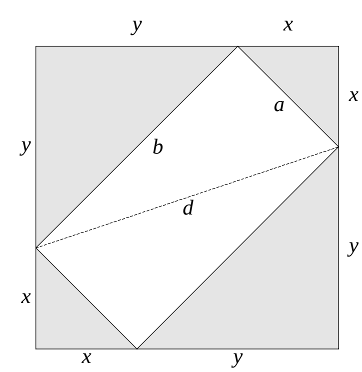
Thus

$$N = 7[3 + 20(2+3t)] = 7(43 + 60t)$$

Then $t = 0$ is the only value that makes N less than 500. Thus $N = 7 \times 43 = 301$.

5. Let x and y be the lengths of the sides of the two isosceles triangles removed from the square, and let a and b be the sides of the rectangle that remains. Then the areas of the triangles removed are $\frac{1}{2}x^2$, $\frac{1}{2}x^2$, $\frac{1}{2}y^2$ and $\frac{1}{2}y^2$. So the total area removed is:

$$x^2 + y^2 = 200.$$



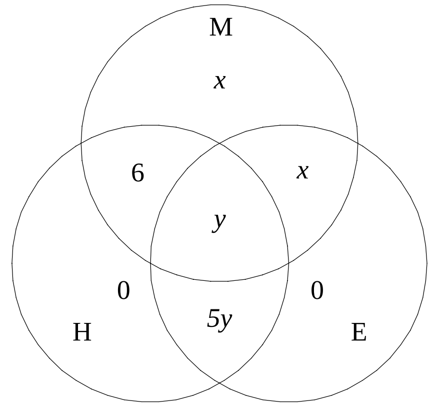
The lengths of the sides of the rectangle are $a = \sqrt{2}x$ and $b = \sqrt{2}y$. Thus the diagonal of the rectangle is given by

$$d^2 = a^2 + b^2 = \left(\sqrt{2}x\right)^2 + \left(\sqrt{2}y\right)^2 = 2\left(x^2 + y^2\right) = 400.$$

Thus $d = \sqrt{400} = \boxed{20}$.

Final Round – Senior Part A

1. Let x be the number of students taking Mathematics only and let y be the number taking all three subjects. In the Venn diagram shown M represents set of students taking Mathematics, E represents those taking English and H represents those taking History. From the diagram we have



$$x + x + y + 5y + 6 = 28 \Rightarrow 2x + 6y = 22 \Rightarrow x + 3y = 11.$$

The only positive integral solutions to this equation are $x = 2, y = 3$, $x = 5, y = 2$ and $x = 8, y = 1$. For the number of students taking History and English only, which is $5y$, to be even y must be even. The only solution satisifying this requirement is $x = 5, y = 2$. Thus the number of students taking Mathematics and English only is $\boxed{5}$.

. . . . (A)

2. For each of the three circles shown the circle to be drawn can be tangent on the inside or the outside. Thus for each circle shown there are two possibilities, so the number of circles that can be drawn tangent to the three circles shown is $2^3 = \boxed{8}$.

. . . . (E)

3. Solving the equation $\log_2(\cos(x)) = -\frac{1}{2}$ for x gives

$$\cos(x) = 2^{-1/2} = \frac{1}{\sqrt{2}} \Rightarrow x = \pm\frac{\pi}{4}, \pm\frac{7\pi}{4}, \pm\frac{9\pi}{4}, \pm\frac{15\pi}{4}, \ldots$$

Thus a possible value for x is $\boxed{\frac{7\pi}{4}}$.

. . . . (E)

4. See Problem 9 from Part A of the Junior Final Round.

. . . . (C)

5. Label the students A, B, C, D and E and assume that they were seated originally at the table in this order. After the break there are $5! = 120$ ways to assign the students to seats at the table. If each student is to have the same neighbors the order must be A, B, C, D, E or E, D, C, B, A, but the first student to be seated, say A can sit in any one of the five seats. Thus there are $5 \times 2 = 10$ ways in which the students can be seated in such a way that they have the same neighbors as before. Thus the probability of all five having the same neighbors is $\frac{10}{120} = \boxed{\frac{1}{12}}$.

Alternate solution: Let A take her place at the table first. Then her new neighbors can be chosen in $\binom{4}{2} = 6$ ways, only one of which will give A the same neighbors. Once A's neighbors are seated, there are two ways to seat the remaining two students, only one of which will result in everyone having the same neighbors. Thus the probability of all five having the same neighbors is $\frac{1}{6} \times \frac{1}{2} = \boxed{\frac{1}{12}}$.

. . . . (E)

6. Obviously any one of A, B or C can win the championship, and neither G nor H have a chance. Players A, B and C must play eight games against the other two for a total of twelve games altogether among the three of them. Each of these twelve games must be won by one of A, B or C. If A wins three of the twelve, B wins four, and C wins five and they all lose all of their other games, then they will be tied at 95 games. Then any player within 28 games of 95 wins, i.e., if the player has already won at least 67 games, can at least tie for the championship. Thus D and E have a chance of at least a tie, but F does not. Thus there are $\boxed{5}$ players who can at least tie for the championship.

.... (C)

7. The error in the approximation is

$$\begin{aligned}\left|1-y-\frac{1}{1+y}\right| &= \left|\frac{(1-y)(1+y)-1}{1+y}\right| = \left|\frac{1-y^2-1}{1+y}\right| = \left|\frac{-y^2}{1+y}\right| \\ &= \left|\frac{y^2}{1+y}\right| = \frac{y^2}{1+y} \qquad \text{since } |y|<1\end{aligned}$$

Then the ratio of the error to the corrrect value is:

$$\frac{\left(\dfrac{y^2}{1+y}\right)}{\left(\dfrac{1}{1+y}\right)} = \left(\frac{y^2}{1+y}\right)(1+y) = \boxed{y^2}$$

.... (B)

8. The triangle $(5, 12, 13)$ is a right triangle, since $5^2 + 12^2 = 13^2$. If we consider the base of all of the triangles to be 12, then for the other four triangles the length of the longest side is changed with the base staying the same. This reduces the altitude to a value less than 5. Thus $\boxed{(5, 12, 13)}$ has the largest area.

.... (B)

9. Since m has two prime divisors, say p and q, we have $m = p^\alpha q^\beta$, for two positive integers α and β. Every divisor of m is a product of a divisor of p^α and a divisor of q^β. This means that the number of divisors m is the product of the number of divisors of p^α and the number of divisors of q,which are $\alpha+1$ and $\beta+1$, respectively. Thus the number of divisors of m is $(\alpha+1)(\beta+1)$. Now $m^2 = p^{2\alpha}q^{2\beta}$, and, as argued above, the number of divisors of m^2 is $(2\alpha+1)(2\beta+1)$. Hence

$$(2\alpha+1)(2\beta+1) = 35 = 5 \times 7$$

and, since α and β are positive integers we must have $2\alpha+1 = 5$ and $2\beta+1 = 7$, or vice versa. Thus $\alpha = 2$ and $\beta = 3$ and $m = p^2q^3$. Thus the number of divisors of m is $(2+1)(3+1) = 3 \times 4 = \boxed{12}$.

.... (A)

10. Let t be the number of minutes since 3:00 when the two hands are next perpendicular. This time will be sometime after 3:30 so we expect t to be greater than 30. The minute hand will have moved through an angle of $\frac{t}{60} \times 360^\circ = (6t)^\circ$. The hour hand will have moved $\frac{t}{60} \times 30^\circ = \left(\frac{t}{2}\right)^\circ$, so the hour hand is $\left(\frac{t}{2}+90\right)^\circ$ from 12:00. When the two hands are next perpendicular we

$$6t - \left(\frac{t}{2}+90\right) = 90 \Rightarrow 6t - \frac{t}{2} = 180 \Rightarrow 11t = 360 \Rightarrow t = \frac{360}{11} = 32\frac{8}{11} \text{ minutes}$$

Express $\dfrac{8}{11}$ minutes in seconds:

$$\frac{8}{11} \text{ minutes} = \frac{8}{11} \times 60 = \frac{480}{11} = 43\frac{7}{11} \text{ seconds}$$

This is closer to 44 than to 43, so second hand is closest to $\boxed{44}$ seconds when the hands are next perpendicular.

.... (D)

Final Round – Senior Part B

1. Let the four terms of the arithmetic sequence be $p = a-3$, $q = a-1$, $r = a+1$, $s = a+3$. Then

$$\begin{aligned} pqrs + 16 &= (a-3)(a-1)(a+1)(a+3) = \left(a^2-9\right)\left(a^2-1\right) + 16 \\ &= a^4 - 10a^2 + 9 + 16 = a^4 - 10a^2 + 25 \\ &= \left(a^2-5\right)^2 \end{aligned}$$

which obviously is a perfect square.

Alternate solution: Let the four term of the arithmetic sequence be p, $q = p+2$, $r = p+4$, $s = p+6$. Then

$$\begin{aligned} pqrs + 16 &= p(p+2)(a+4)(p+6) + 16 = [p(p+6)][(p+2)(p+4)] + 16 \\ &= \left(p^2+6p\right)\left(p^2+6p+8\right) + 16 \end{aligned}$$

Set $A = p^2 + 6p$, then

$$pqrs + 16 = A(A+8) + 16 = A^2 + 8A + 16 = (A+4)^2 = \left(p^2+6p+4\right)^2$$

which is again a perfect square.

2. Possible sequences that give 1999 are $\cdots 999\underbrace{1\,999}2\cdots$, $\cdots 99\underbrace{19\,99}20\cdots$, $\cdots 9\underbrace{199\,9}200\cdots$. The last one is obviously the first time that 1999 appears in the sequence.

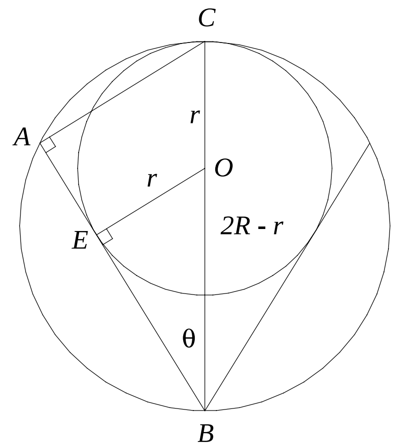
a. Thus the string contains

$$4\,[9199 - 2000 + 1] + 1 = \boxed{22801} \text{ digits}$$

b. Between 2000 and 8999, inclusive, there are 7000 integers, each of which can only have the digit 0 in the final three positions. There are 21000 possible digits where 0 could occur and 0 occurs in $\frac{1}{10}$ of them, for a total of 2100 zeros between 2000 and 8999. For the 100 integers between 9000 and 9099, each integer has a 0 in the hundreds position, and, in addition, 0's occur in $\frac{1}{10}$ of the 200 digits in the units and tens positions for a total of $100 + 20 = 120$ zeros. Between 9100 and 9199 zeros occur in $\frac{1}{10}$ of the 200 digits in the units and tens postions for a total of 20 zeros. The grand total is thus $2100 + 120 + 20 = \boxed{2240}$.

3. The factors of 1000 can be paired as: 1×10000, 2×5000, 4×2500, 5×2000, 8×1250, 10×1000, 16×625, 20×500, 25×400, 40×250, 50×200, 80×125, with 100 left by itself. The sum of the logarithms is equal to the logarithm of the product, thus

$$\begin{aligned} \log 1 + \log 2 + \log 4 + \cdots &+ \log 5000 + \log 10000 = \log\left(1 \times 2 \times 4 \times \cdots \times 5000 \times 10000\right) \\ &= \log\left(10000^{12} \times 100\right) = \log\left(10^{48} \times 10^2\right) \\ &= \log\left(10^{50}\right) = \boxed{50} \end{aligned}$$



4. Since BC is a diameter of the large circle $\angle CAB$ is a right angle, and since AB is tangent to the small circle $\angle OEB$ is a right angle. Thus $\triangle ABC$ and $\triangle EBO$ are similar triangles, so that

$$\frac{CA}{CB} = \frac{OE}{OB} \Rightarrow \frac{CA}{2R} = \frac{r}{2R-r} \Rightarrow CA = 2R\left(\frac{r}{2R-r}\right)$$

Thus

$$\begin{aligned} AB^2 &= BC^2 - CA^2 = (2R)^2 - \frac{4R^2r^2}{(2R-r)^2} \\ &= 4R^2\left[\frac{4R^2-4Rr+r^2-r^2}{(2R-r)^2}\right] = 16R^2\left[\frac{R^2-Rr}{(2R-r)^2}\right] \end{aligned}$$

So that

$$AB = \frac{4R\sqrt{R^2-rR}}{2R-r}$$

Alternate solution: Let $\angle ABC = \theta$. Then $\sin\theta = \dfrac{r}{2R-r}$ and

$$AB = 2R\cos\theta = 2R\sqrt{1-\sin^2\theta} = 2R\sqrt{1-\frac{r^2}{(2R-r)^2}} = 2R\frac{\sqrt{4R^2-4Rr+r^2-r^2}}{2R-r} = \frac{4R\sqrt{R^2-Rr}}{2R-r}$$

5. Let x be the number of litres of fuel in the tank at the beginning of the cycle as listed. Note that the number of litres that flow into the tank during a 24 hour period is $2000 \times 24 = 48,000$ and the amount that is removed during the same period is $6,000 + 13,500 + 7,300 + 10,000 + 8,000 + 3,200 = 48,000$, so that the tank is in equilibrium. Further, note that the amount that flows into the tank during any four hour period is 8000 litres. Consider the following table which shows the amount of fuel in the tank at the beginning of each of the four hour periods.

Time (hr)	Amount in tank (litres)	Changes inflow – outflow (litres)
0	x	$8000 - 6000 = 2000$
4	$x + 2000$	$8000 - 13500 = -5500$
8	$x - 3500$	$8000 - 7300 = 700$
12	$x - 2800$	$8000 - 10000 = -2000$
16	$x - 4800$	$8000 - 8000 = 0$
20	$x - 4800$	$8000 - 3200 = 4800$
24	x	

Since the minimum amount of fuel in the tank is 200 litres and from the table above we see that this amount is $x - 4800$, we must have $x - 4800 \geq 200$ or $x \geq 5000$. Further, since the maximum amount of fuel in the tank is $x + 2000$ the tank must have a capacity of $x + 2000 = 7000$ litres. If the cycle is started at the beginning or end of the time period when 8000 litres is removed, we can start with 200 litres. However, the capacity of the tank must still be 7000 litres.