BRITISH COLUMBIA COLLEGES
HIGH SCHOOL MATHEMATICS CONTEST
2000 SOLUTIONS

Junior Preliminary

1. Let x be the number of litres of gasoline in the tank prior to filling. Then $x + 15 = \frac{3}{4} \cdot 28$, or $x = 6$.
Answer is (D)

2. The first figure is composed of one square of side 1 (consisting of 4 matchsticks) plus 2 squares of side 1 each missing 1 matchstick, for a total of $4 + 2 \cdot 3 = 10$ matchsticks. Each subsequent figure consists of the previous figure plus 2 squares of side 1 each missing 1 matchstick. Thus the n^{th} figure in the sequence contains $4 + 2 \cdot 3 \cdot n = 6n + 4$ matchsticks. The largest value n for which 500 matchsticks is sufficient is thus 82 (which uses up $6 \cdot 82 + 4 = 496$ matchsticks). Now the number of squares in the first figure is 3 and each subsequent figure contains 2 more squares than the previous one. Therefore the number of squares in the n^{th} figure is $2n + 1$. For $n = 82$ this means that 165 squares would be in the largest figure made with 500 matchsticks.
Answer is (B)

3. Let ℓ and w be the length and width (in metres) of the rectangle in question. Since the perimeter is 56 metres, we have $2\ell + 2w = 56$, or $\ell + w = 28$. We are also told that $\ell : w = 4 : 3$, or $\ell = \frac{4}{3}w$. Using this in the first equation we get
$$\begin{aligned} \tfrac{4}{3}w + w &= 28 \\ \tfrac{7}{3}w &= 28 \\ w &= 12 \end{aligned}$$
which implies that $\ell = 16$. By the Theorem of Pythagoras the length of the diagonal is
$$\sqrt{12^2 + 16^2} = \sqrt{400} = 20.$$
Answer is (B)

4. Since there are 31 days in March, there are 31 days between March 23 and April 23. That is, the period in question is 4 weeks and 3 days. Since April 23 is a Tuesday, we must have March 23 a Saturday, namely 3 days earlier in the week.
Answer is (A)

5. The total area of the board is $25x^2$ square units. The area of the shaded region is $x \cdot 4x + x \cdot 3x = 7x^2$ square units. Therefore, the probability of hitting the shaded area is
$$\frac{7x^2}{25x^2} = \frac{7}{25} = 0.28$$
Answer is (D)

6. Let us compute the sum of the proper divisors of each of the 5 possible answers in the list:
$$\begin{aligned} 13 &: \quad 1 < 13 \\ 16 &: \quad 1 + 2 + 4 + 8 = 15 < 16 \\ 30 &: \quad 1 + 2 + 3 + 5 + 6 + 10 + 15 = 42 > 30 \\ 44 &: \quad 1 + 2 + 4 + 11 + 22 = 40 < 44 \\ 50 &: \quad 1 + 2 + 5 + 10 + 25 = 43 < 50 \end{aligned}$$
The only one of these which qualifies as an abundant number is 30.
Answer is (C)

7. The area in question is the area of a trapezoid less the area of a semicircle. The area of the semicircle is obviously $\frac{1}{2}\pi 2^2 = 2\pi$ cm^2. The area of the trapezoid is $\frac{1}{2}4(4 + 6) = 20$ cm^2. Thus the shaded area is $20 - 2\pi$ cm^2.
Answer is (D)

8. Let the coordinates of the point S be (x, y). Since $PS \parallel QR$ they must have the same slope:

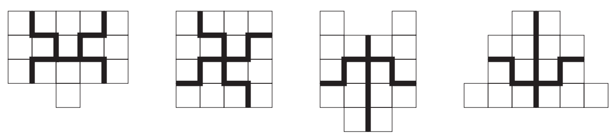
$$\frac{y+2}{x+3} = \frac{-5-1}{1-9} = \frac{3}{4}$$
$$\text{or} \qquad 4y - 3x = 1$$

Since $RS \parallel QP$ we also have (by the same argument):

$$\frac{y-1}{x-9} = \frac{-5+2}{1+3} = -\frac{3}{4}$$
$$\text{or} \qquad 4y + 3x = 31$$

From these two equations in 2 unknowns we easily solve for $x = 5$ and $y = 4$. Thus $x + y = 9$.
Answer is (E)

9. The diagram below shows how figures (a), (c), (d), and (e) can be filled with copies of the "T" tile. No matter how one tries figure (b) cannot be filled with copies of it.



(a) (c) (d) (e)

Answer is (B)

10. Note first that

$$2^{5555} = (2^5)^{1111} = 32^{1111}$$
$$3^{3333} = (3^3)^{1111} = 27^{1111}$$
$$6^{2222} = (6^2)^{1111} = 36^{1111}$$

Since $27 < 32 < 36$, we have $27^{1111} < 32^{1111} < 36^{1111}$, which means

$$3^{3333} < 2^{5555} < 6^{2222}$$

Answer is (E)

11. Let us first compute the number of seconds in 1 day, 1 hour, 1 minute, and 1 second, and then multiply by 2000. Now 1 day plus 1 hour is clearly 25 hours. Then 1 day, 1 hour, plus 1 minute is $25 \times 60 + 1 = 1501$ minutes. Expressed in seconds this is $1501 \times 60 = 90060$ seconds. Thus 1 day, 1 hour, 1 minute, and 1 second is 90,061 seconds. The answer to the problem is this figure multiplied by 2000; that is, 180,122,000, which to the nearest million is 180,000,000.
Answer is (D)

12. If $a = b = c$, then $100a + 10b + c = 100a + 10a + a = 111a$. Since a can be any digit, in order for a number to be a factor of the three-digit number, it must be a factor of 111. The factors of 111 are 1, 3, 37, and 111. The only one of these appearing in the list is 37.
Answer is (E)

13.
$$\begin{aligned}(x+y)^2 - (x-y)^2 > 0 \quad &\Leftrightarrow \quad x^2 + 2xy + y^2 - x^2 + 2xy - y^2 > 0 \\ &\Leftrightarrow \quad 4xy > 0 \\ &\Leftrightarrow \quad xy > 0\end{aligned}$$

The last condition clearly holds if and only if x and y have the same sign; that is, both are positive or both are negative.
Answer is (A)

14. Let a, b, c be the three sides of the triangle. Let us assume that $a \le b \le c$. Since the perimeter is 12, we have $a + b + c = 12$. Let us now list all possible sets of integers (a, b, c) satisfying the above conditions:

$$(1,1,10),\quad (1,2,9),\quad (1,3,8),\quad (1,4,7),\quad (1,5,6),\quad (2,2,8),$$
$$(2,3,7),\quad (2,4,6),\quad (2,5,5),\quad (3,3,6),\quad (3,4,5),\quad (4,4,4)$$

However, it is clear that some of these "triangles" do not actually exist, since in any triangle the sum of the lengths of the two shorter sides must be greater than the length of the longest side. With this additional condition we have only the following triangles (a, b, c):

$$(2,5,5),\quad (3,4,5),\quad (4,4,4)$$

We can now examine the 4 statements and conclude that (i), (ii) and (iv) are clearly true. As for (iii), we see that triangle $(3, 4, 5)$ above is right-angled; hence (iii) is false. Answer is (D)

15. The area of the original triangle is $\frac{1}{2}bh$. The new triangle has altitude $h + m$ and base $b - x$. We need to find x such that the area of the new triangle is $\frac{1}{4}bh$. Clearly the area of the new triangle is $\frac{1}{2}(h + m)(b - x)$. Thus

$$\begin{aligned} \tfrac{1}{4}bh &= \frac{1}{2}(h+m)(b-x) \\ \frac{bh}{2(h+m)} &= b - x \\ x &= b - \frac{bh}{2(h+m)} = \frac{2bh + 2bm - bh}{2(h+m)} = \frac{b(2m+h)}{2(h+m)} \end{aligned}$$

Answer is (E)

Senior Preliminary

1. Antonino averages 15 km/h for the first 20 km. This means it takes him $20/15 = 4/3$ hours to cover the first 20 km. In order to average 20 km/h for a 40 km distance, he must cover the distance in 2 hours. He only has 2/3 h remaining in which to cover the last 20 km. His speed over this last 20 km then must be (on average) $20/(2/3) = 30$ km/hr. Answer is (B)

2. Let C be the centre of the circle. Since the points O and B are equidistant from the centre of the circle and also equidistant from the point P, and since both O and B lie on the x-axis, we see that P has coordinates $(3, y)$ with $y < 0$. The slope of OC is $2/3$. Since PO is the tangent line to the circle at O we know that $PO \perp OC$. Therefore the slope of PO is $-3/2$. However, the slope of PO is computed to be $(y - 0)/(3 - 0) = y/3$. Together these imply that $y = -9/2$. Answer is (C)

3. First of all the number of possible ways to choose a pair of distinct students from a set of five is $\binom{5}{2} = \frac{5!}{2!3!} = 10$. From this we need only eliminate those whose age difference is 1. Clearly there are exactly 4 such, namely $(6, 7)$, $(7, 8)$, $(8, 9)$, and $(9, 10)$. So our probability of success is 6 out of 10, or 3/5. Answer is (C)

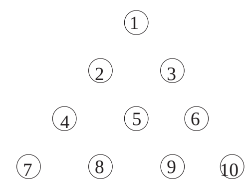
4. The straight sections of the belt are tangent to all 3 pulleys and thus perpendicular to the radius of each pulley at the point of contact. Thus the straight sections of the belt are the same lengths as the distances between the centres of the pulleys, which are 5, 12, and $\sqrt{5^2 + 12^2} = 13$; so the straight sections of belt add up to 30 units. The curved sections of belt, when taken together, make up one complete pulley, or a circle of radius 2. Thus the curved sections add up to $2\pi(2) = 4\pi$. So the full length of the belt is $30 + 4\pi$ units. Answer is (B)

5. Let n be the number of quizzes Mark has already taken. Let x be his total score on all n quizzes. Then we have the following:

$$\frac{x+71}{n+1} = 83 \quad \Rightarrow \quad x = 83(n+1) - 71$$
$$\frac{x+99}{n+1} = 87 \quad \Rightarrow \quad x = 87(n+1) - 99$$

Solving this system yields $n = 6$. Answer is (C)

6. Number the pins as shown in the diagram on the right. There is then 1 large equilateral triangle with 4 pins on a side, namely the one with vertices numbered $(1,7,10)$. There are also 3 equilateral triangles with 3 pins on a side, namely the ones whose vertices are numbered $(1,4,6)$, $(2,7,9)$, and $(3,8,10)$. The equilateral triangles with 2 pins on a side come in 2 distinct orientations, one with a single vertex above the horizontal base and one with a single vertex below the horizontal base. For the first type we have 6 such: $(1,2,3)$, $(2,4,5)$, $(3,5,6)$, $(4,7,8)$, $(5,8,9)$, and $(6,9,10)$. For the second type we have only 3 such: $(2,3,5)$, $(4,5,8)$, and $(5,6,9)$. This gives us a total of 13 equilateral triangles so far. However, there are two others which are skewed somewhat to the edges of the outer triangle: $(2,6,8)$ and $(3,4,9)$, which gives us a total of 15 equilateral triangles. Answer is (A)

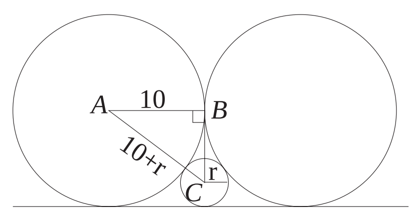


7. The critical idea here is to recognize that when the square covers as much of the triangle as possible, the triangle will also cover as much of the square as possible, and that at this point the amount of triangle covered is the same as the amount of square covered. Let A be the area of the triangle. Then $0.6A = \frac{2}{3} \cdot 36$, or $A = 40 \text{ cm}^2$. Answer is (D)

8. Let A be the centre of one of the large circles, let B be the point of contact between the two large circles and let C be the centre of the small circle. Then $AB \perp BC$, $AC = 10 + r$ and $BC = 10 - r$. From the Theorem of Pythagoras we have

$$\begin{aligned}
(10+r)^2 &= 10^2 + (10-r)^2 \\
100 + 20r + r^2 &= 100 + 100 - 20r + r^2 \\
40r &= 100 \\
r &= 2.5
\end{aligned}$$

Answer is (B)



9. See #10 on the Junior paper. Answer is (E)

10. Clearly y must be even in order to get integer solutions. The largest possible value for y is 16 since we must have $x > 0$. When $y = 16$ we have $x = 1$, so $(x,y) = (1,16)$ is a solution. Let us consider successively smaller (even) values for y: $(x,y) = (4,14)$, $(7,12)$, $(10,10)$, etc. However, the solution $(10,10)$ and any further ones do not satisfy $y > x$. So we are left with the solutions $(x,y) = (1,16)$, $(4,14)$, and $7,12)$. Answer is (E)

11. Since B and C are positive integers, we see that $B + 1/(C+1) > 1$, whence its reciprocal is smaller than 1. Therefore, A must represent the integer part of $24/5$, i.e. $A = 4$. Then we have

$$\frac{4}{5} = \frac{1}{B + \dfrac{1}{C+1}} \qquad \text{or} \qquad \frac{5}{4} = B + \frac{1}{C+1}$$

For exactly the same reason as above we see that B must be the integer part of $5/4$, i.e. $B = 1$. Then $1/4 = 1/(C+1)$, which implies that $C = 3$. Then

$$A + 2B + 3C = 4 + 2(1) + 3(3) = 15.$$

Answer is (C)

12. The number of ways of choosing 2 balls (without replacement) from a box with $m+n$ balls is $\binom{m+n}{2}$. The number of ways of choosing 1 white ball is m, and the number of ways of choosing 1 black ball is n. Thus the probability that one ball of each colour is chosen is:

$$\frac{mn}{\binom{m+n}{2}} = \frac{mn}{\frac{(m+n)!}{2!(m+n-2)!}} = \frac{mn}{\frac{(m+n)(m+n-1)}{2}} = \frac{2mn}{(m+n)(m+n-1)}$$

Answer is (D)

13. Since x builders build z houses in y days, we see that x builders build z/y houses per day, or 1 builder builds $z/(xy)$ houses per day. Hence q builders build $qz/(xy)$ houses per day. Let s be the number of days needed for q builders to build r houses. Then q builders build r/s houses per day. Equating these we get $qz/(xy) = r/s$, whence $s = rxy/(qz)$ days. Answer is (D)

14. Adding the two given equations together we get:

$$\begin{aligned} x^2 + 2xy + y^2 + x + y &= 42 \\ \text{i.e.} \quad (x+y)^2 + (x+y) - 42 &= 0 \\ (x+y-6)(x+y+7) &= 0 \end{aligned}$$

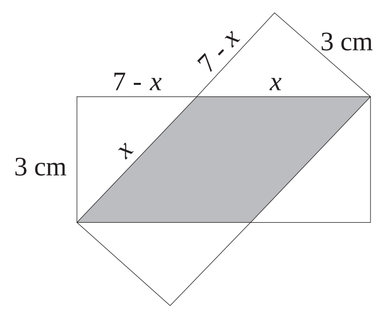
Thus $x + y = 6$ or $x + y = -7$. Answer is (A)

15. All the unshaded triangles in the diagram below are right-angled and thus are congruent. By the Theorem of Pythagoras we have

$$\begin{aligned} x^2 &= (7-x)^2 + 3^2 \\ &= 49 - 14x + x^2 + 9 \\ 14x &= 58 \\ x &= \tfrac{29}{7} \end{aligned}$$

The area of the shaded parallelogram is $3x = \frac{87}{7}$ cm^2.

Answer is (A)



Junior Final

Part A

1. If we square the digits from 0 to 9 and consider the final digit of the square we get only the digits 0, 1, 4, 9, 6, and 5. Since there are no others, we see that 8 is NOT the final digit of any square.

Answer is (E)

2. Each of the 10 straight lines intersects each of the others exactly once. This makes for 90 intersections; however, each of the intersections is counted twice in this approach, depending upon which of the two lines we consider first. To get the correct number of intersections we simply divide 90 by 2 to get 45.

Answer is (A)

3. Let us try successively to make up each of the given amounts using 6 coins:

$$\begin{aligned}91 &= 1+5+10+25+25+25\\87 &= 1+1+10+25+25+25\\78 &= 1+1+1+25+25+25\\51 &= 1+10+10+10+10+10\end{aligned}$$

Thus each of the first 4 choices can be made up with 6 coins. In order to make up 49 we would need to use 4 pennies. This would require us to make up the total of 45 cents with only 2 coins, which is clearly impossible. Answer is (E)

4. First observe that

$$(x-y)^2 = x^2+y^2-2xy = 28-2(14) = 0$$

This means that $x-y=0$, i.e. $x=y$. In that event we clearly have $x^2-y^2=0$. Answer is (B)

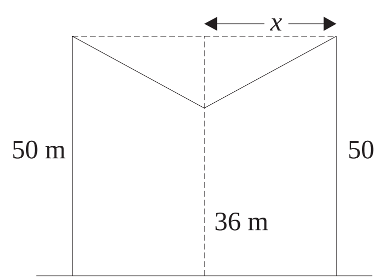
5. See #10 on Senior Preliminary above. Answer is (E)

6. We are seeking the largest integer n such that

$$\frac{n(n+1)}{2} \le 500 \qquad \text{or} \qquad n(n+1) \le 1000$$

Since $32^2 = 1024$ we see that $n < 32$. Checking $n = 31$ we find $31\cdot 32 = 992$. Thus the integer n we seek is 31. The triangular number associated with this value of n is $\frac{1}{2}(992) = 496$. Answer is (C)

7. In order for the rope to be at the lowest possible point, that point must be the middle of the rope. Thus we are faced with solving a right-angled triangle with hypotenuse 40 m and one side of $50-36=14$ m. By the Theorem of Pythagoras the third side (x in the diagram at the right) is $\sqrt{40^2-14^2} = \sqrt{1404} = 6\sqrt{39}$. The distance between the two flagpoles is $2x = 12\sqrt{39}$.
Answer is (C)



8. Let a_n be the number of people who arrive at the n^{th} ring of the door bell. Then $a_n = 2n-1$. Let b_n be the number of people who have arrived after the n^{th} ring of the door bell. Then we have

$$\begin{aligned}b_1 &= 1\\b_{n+1} &= b_n + a_{n+1} \qquad \text{for } n \ge 1\\&= b_n + 2n+1\end{aligned}$$

This can be rewritten as

$$\begin{aligned}b_1 &= 1\\b_{n+1} - b_n &= 2n+1 \qquad \text{for } n \ge 1\end{aligned}$$

If we write out the first 20 of these we get

$$\begin{aligned}a_1 &= 1\\b_2 - b_1 &= 3\\b_3 - b_2 &= 5\\&\vdots\\b_{20} - b_{19} &= 39\end{aligned}$$

When we add all 20 of the above equations together we get

$$b_{20} = 1 + 3 + 5 + \cdots + 39 = \tfrac{1}{2}20\cdot(2\cdot1 + 19\cdot2) = 400$$

where we have used the well-known formula for the sum of an arithmetic progression with n terms, having first term a and common difference d: $\frac{1}{2}n(2a + (n-1)d)$. Answer is (E)

9. According to the definition of the operation $*$ we have

$$2 * (-1) = 2^{-1} - (-1)^2 = \tfrac{1}{2} - 1 = -\tfrac{1}{2}$$

Answer is (C)

10. Let a be the length of PQ, QR, and RS. Then the radii of the 3 circles are a, $2a$, and $3a$. The area between the inner and middle circles is then $\pi(2a)^2 - \pi a^2 = 3\pi a^2$, and the area between the middle and outer circles is $\pi(3a)^2 - \pi(2a)^2 = 5\pi a^2$. Thus the ratio we want is $3\pi a^2 : 5\pi a^2 = \frac{3}{5}$. Answer is (D)

Part B

1. (a) For this part of the question, the simplest method is simply to list all the possible numbers. In increasing order they are:

   111, 112, 113, 122, 123, 133, 222, 223, 233, and 333

   for a total of 10 numbers.

   (b) Again, most junior students will simply try to list all the possible integers. In increasing order they are:

   1111, 1112, 1113, 1114, 1122, 1123, 1124, 1133, 1134, 1144, 1222, 1223, 1224, 1233,
   1234, 1244, 1333, 1334, 1344, 1444, 2222, 2223, 2224, 2233, 2234, 2244, 2333, 2334,
   2344, 2444, 3333, 3334, 3344, 3444, and 4444

   for a total of 35 numbers.

   A more sophisticated approach (which can be generalized) follows: We first define $n(k, d)$ to be the number of k-digit integers ending with the digit d and satisfying the two conditions (i) and (ii) in the problem statement. Since a k-digit number ending with the digit d consists of appending the digit d to all $(k-1)$-digit numbers ending with a digit less than or equal to d, we have

   $$n(k, d) = n(k-1, 1) + n(k-1, 2) + \cdots + n(k-1, d) \qquad (*)$$

   Furthermore, we also have $n(1, d) = 1$ for all digits d and $n(k, 1) = 1$ for all integers k. The relationship $(*)$ allows us to create the following table of values for $n(k, d)$:

$k\backslash d$	1	2	3	4
1	1	1	1	1
2	1	2	3	4
3	1	3	6	10
4	1	4	10	20

   Each entry in the table is the sum of the entries in the previous row up to and including the column containing the given entry (note the presence of Pascal's Triangle in the table). From the answer to parts (a) and (b) are:

   $$\text{(a)}: n(3,1) + n(3,2) + n(3,3) = 1 + 3 + 6 = 10$$
   $$\text{(b)}: n(4,1) + n(4,2) + n(4,3) + n(4,4) = 1 + 4 + 10 + 20 = 35$$

   Clearly this table could have been extended to deal with any number k and with any digit $d \leq 9$.

2. Since $ABCD$ is a square the lines AC and BD are perpendicular. Since the circle had radius 1 unit, the Theorem of Pythagoras tells us that $AB = BC = CD = DA = \sqrt{2}$. The tangent PC at C is perpendicular to the diameter AC; thus $\angle PCB = 45°$. Since $PA \perp BC$ we also have $\angle CPB = 45°$. This makes $\triangle PBC$ isosceles, which means that $PB = BC = \sqrt{2}$. Applying the Theorem of Pythagoras to $\triangle APD$ we have
$$PD^2 = AP^2 + AD^2 = (2\sqrt{2})^2 + \sqrt{2}^2 = 8 + 2 = 10$$
from which we see that $PD = \sqrt{10}$.

3. Since the 45×30 rectangle has its sides in the proportion $3 : 2$ we will consider first looking at a 3×2 rectangle, in which there are 2 vertices which lie on the diagonal. In the original 45×30 rectangle we need only consider the fifteen 3×2 rectangle which straddle the diagonal in question. The lower left of these has its lower leftmost vertex on the diagonal, and each of these 3×2 rectangles adds a further vertex to the count for its upper rightmost corner. This gives us a total of $1 + 15 = 16$ vertices on the diagonal.

4. (a) We will use the proof in the problem statement as a model. Consider $bc - ad$. Clearly $(bc - ad)^2 \geq 0$. Expanding gives
$$b^2c^2 - 2abcd + a^2d^2 \geq 0$$
This is easily rearranged to yield $b^2c^2 + a^2d^2 \geq 2abcd$.

   (b) We will use part (a) to prove part (b). Since a, b, c, and d are arbitrary real numbers, the inequality in part (a) remains true for any rearrangement of the letters: in particular we have:
$$2abcd \leq b^2c^2 + a^2d^2$$
$$2abdc \leq b^2d^2 + a^2c^2$$
$$2adcb \leq d^2c^2 + a^2b^2$$
Recognizing that multiplication is commutative for real numbers we can reorganize the products in each of the above inequalities and sum the three inequalities to get the desired result.

5. (a) Let us place 2 (outer) coins next to the original coin so that they touch each other. Then the centres of the 3 coins form an equilateral triangle with side length equal to twice the radius of a single coin. Therefore the angle between the centres of the 2 (outer) coins measured at the centre of the first coin is 60°. Since 6 such angles make up a full revolution around the inner coin, we can have exactly 6 outer coins each touching the original (inner) coin and also touching its other two neighbours.

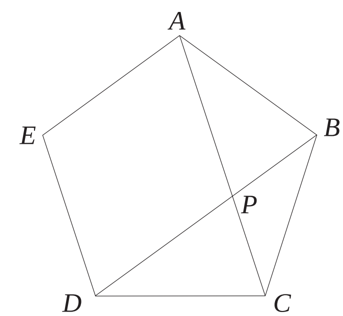
   (b) There are 6 non-overlapping spaces whose areas we must add; each is found between 3 coins which simultaneously touch other, and whose centres form the equilateral triangle mentioned in part (a) above. This equilateral triangle has side length 2, since we are given the radii of the coins as 1. Our strategy to compute the area of one such space is to find the area of the equilateral triangle and subtract the areas of the 3 circular sectors found within the triangle. The altitude of the equilateral triangle with side length 2 can be easily found (Theorem of Pythagoras) as $\sqrt{3}$. Thus the area of the triangle itself is $\frac{1}{2} \cdot 2 \cdot \sqrt{3} = \sqrt{3}$. The area of a single coin is $\pi \cdot 1^2 = \pi$. The circular sectors within the equilateral triangle are each one-sixth of the area of the coin; there are 3 such sectors which gives us a total area of one-half the area of a single coin to be subtracted from the area of the equilateral triangle. Thus the area of a single space is $\sqrt{3} - (\pi/2)$. Since there are 6 such spaces, we have a total area of $6\sqrt{3} - 3\pi$ square units.

Senior Final

Part A

1. The area of the trapezoid $ABCD$ is $\frac{1}{2}\cdot 4\cdot(1+10) = 22$ square units. We want to choose the point P such that the area of $\triangle CPB$ is half this area, i.e. 11 square units. Let the base PB have length x. Then the area of $\triangle CPB = \frac{1}{2}\cdot 4\cdot x = 2x$. Since this must be 11, we have $x = 5\frac{1}{2}$. Answer is (E)

2. Let P be the intersection of AC and BD as in the diagram below. The sum of the interior angles of a (regular) pentagon is $(5-2)180° = 540°$. So each interior angle in a regular pentagon has measure $540°/5 = 108°$. Since $\triangle ABC$ is isosceles with vertex angle equal to $108°$. The base angles, namely $\angle BAC$ and $\angle BCA$, both have measure $36°$. Similarly, $\angle CBD = \angle CDB = 36°$. Since $\angle ABC = 108°$ and $\angle CBD = 36°$, we see that $\angle ABP = 108° - 36° = 72°$. This means that the third angle in triangle ABP, namely $\angle APB$ has measure $180° - 36° - 72° = 72°$. Thus the angles at P have measure $72°$ and $180° - 72° = 108°$. Answer is (A)



3. Let a, b, c be the ages (in integers) of the three children. We may assume that $a \le b \le c < 15$. Since the product $abc = 90$, we also know that $a > 0$. The values we seek for a, b, and c are integers which divide evenly into 90 and lie between the values 1 and 14 (inclusive). The only such integers are 1, 2, 3, 5, 6, 9, and 10. We will now look at all possible products which satisfy the above conditions, and for each one we will compute the sum of the ages:

a	b	c	sum
1	9	10	20
2	5	9	16
3	3	10	16
3	5	6	14

We are further told that even knowing the sum of the ages would NOT allow us to determine the three ages. Thus we are forced to conclude that the sum must be 16, since there are two distinct sets of ages which sum to 16 in the above table. The only ages found in these two sets are 2, 3, 5, 9, and 10. We notice that 1 and 6 do not appear. Answer is (D)

4. Let V be the volume of a full tub (in litres, say). Then the rate at which the hot water can fill the tub is $V/10$ litres per minute. Similarly the rate at which the cold water can fill the tub is $V/8$ litres per minute. On the other hand a full tub empties at the rate of $V/5$ litres per minute. If all three are happening at the same time then the rate at which the tub fills is:
$$\frac{V}{10} + \frac{V}{8} - \frac{V}{5} = \frac{4V + 5V - 8V}{40} = \frac{V}{40}$$
litres per minute, which means it takes 40 minutes to fill the tub. Answer is (C)

5. We first need to recall that the sum of the first n integers is given by $n(n+1)/2$. The sum we are presented with is now the difference between the sums of the first $k+19$ integers and the first k integers. Using the above formula we have:
$$\begin{aligned}(k+1)+(k+2)+\cdots(k+19) &= \frac{(k+19)(k+20)}{2} - \frac{k(k+1)}{2} \\ &= \frac{k^2+39k+380-k^2-k}{2} \\ &= \frac{38k+380}{2} = 19(k+10)\end{aligned}$$

Since 19 is prime, in order for $19(k+10)$ to be a perfect square, $k+10$ must contain 19 as a factor. The smallest such value occurs when $k+10=19$, i.e. when $k=9$, and we indeed get a perfect square in this case, namely 19^2. Answer is (B)

6. Let n be the number in question. Then n can be written as 10^5+a where a is a number with at most 5 digits. Moving the left-most digit (the digit 1) to the extreme right produces a number $10a+1$. The information in the problem now tells us that $10a+1=3(10^5+a)=300000+3a$, or $7a=299999$. This yields $a=42857$. So $n=142857$ (and the other number we created is 428571), the sum of whose digits is $1+4+2+8+5+7=27$. Answer is (D)

7. Let us organize this solution by considering the size of the largest cube in the subdivision of the original cube. The largest could have a side of size 4 cm, 3 cm, 2 cm, or 1 cm. In each of these 4 cases we will determine the minimum number of cubes possible. In the first case, when there is a cube of side 4 cm present, we can only include cubes of side 1 cm to complete the subdivision, and we would need 61 such since the cube of side 4 cm uses up 64 cm^3 of the 125 cm^3 in the original cube. Thus, in this case we have 62 cubes in the subdivision. If we look at the other extreme case, namely when the largest cube in the subdivision has side length of 1 cm, we clearly need 125 cubes for the subdivision. We also note here that we will certainly decrease the number of cubes in a subdivision if we try to replace sets of 1×1 cubes by larger cubes whenever possible. Now consider the case when there is a cube of side length 3 cm present. If we place it anywhere but in a corner, the subdivision can only be completed by cubes of side length 1 cm, which gives us $1+98=99$ cubes in total. If we place it in a corner, we can then place 4 cubes of side length 2 cm on one side of the larger cube, 2 more such cube on a second side and a third such cube on the third side; this gives us 1 large cube and 7 medium cubes for a total volume of $27+7(8)=83$ cm^3 which means we still 42 small cubes, for a grand total of 50 cubes. It is easy to see that if the largest cube has side length 2 cm we can place at most 8 of them in the original cube and the remainder of the volume must be made up of cubes of side length 1 cm; this gives a total of $8+61=69$ cubes. Thus the smallest number of cubes possible is 50 and in this case there are 7 cubes of side length 2 cm. Answer is (D)

8. We need to find a sequence of all 9 councillors beginning with A and ending with E such that each pair of consecutive councillors are 'on speaking terms' with each other. When one first looks at the table provided, it looks a little daunting. However, a first observation is that among the councillors other than A and E (who need to appear at the ends of the sequence) councillors F and H are only 'on speaking terms' with 2 others, one of which is councillor B. Thus councillor F must receive the rumor from B and pass it to I, or vice versa. Similarly, councillor H must hear the rumor from B and pass it to C, or vice versa. Thus we must have either $I-F-B-H-C$ or $C-H-B-F-I$ as consecutive councillors in the sequence. Since A and E lie on the ends, and neither of them are 'on speaking terms with' either councillors C or I, we see that councillors D and G must be placed one on either end of the above subsequence of 5 councillors. This leaves us with either $D-I-F-B-H-C-G$ or $G-C-H-B-F-I-D$. Councillors A and E can be placed on the front and rear of either of these sequences to gives the final sequence as either $A-D-I-F-B-H-C-G-E$ or $A-G-C-H-B-F-I-D-E$. In either case the fourth person after councillor A (who started the rumor) to hear the rumor was councillor B. Answer is (A)

9. Let us examine the temperature differences on the respective pairs of thermometers. A difference of 24° on A corresponds to a difference of 16° on B; thus they are in the ratio of $3:2$. A difference of 12° on B corresponds to a difference of 72° on C; thus they are in the ratio of $1:6$. Now a temperature drop of 18° on A means a drop of 12° on B (using the ratio $3:2$). This results in a temperature drop of 72° on C (using the ratio $1:6$). Answer is (E)

10. The number of positive integers less than or equal to n which are multiples of k is the integer part of n/k (that is, perform the division and discard the decimal fraction, if any). This integer is commonly denoted $\lfloor n/k \rfloor$. Thus the number of positive integers between 200 and 2000 which are multiples of 6 is

$$\left\lfloor \frac{2000}{6} \right\rfloor - \left\lfloor \frac{200}{6} \right\rfloor = 333 - 33 = 300$$

Similarly, the number of positive integers between 200 and 2000 which are multiples of 7 is

$$\left\lfloor \frac{2000}{7} \right\rfloor - \left\lfloor \frac{200}{7} \right\rfloor = 285 - 28 = 257$$

In order to count the number of positive integers between 200 and 2000 which are multiples of 6 or 7 we could add the above numbers. This, however, would count the multiples of both 6 and 7 twice; that is, the multiples of 42 would be counted twice. Thus we need to subtract from this sum the number of positive integers between 200 and 2000 which are multiples of 42. That number is

$$\left\lfloor \frac{2000}{42} \right\rfloor - \left\lfloor \frac{200}{42} \right\rfloor = 47 - 4 = 43$$

Therefore, the number of positive integers between 200 and 2000 which are multiples of 6 or 7 is $300 + 257 - 43 = 514$. But we are asked for the number of positive integers which are multiples of 6 or 7, but NOT BOTH. Thus we need to again subtract the number of multiples of 42 in this range, namely 43. The final answer is $514 - 43 = 471$. Answer is (B)

Part B

1. First draw the radius OD. Let $\angle E = \alpha$. Since $DE = r = OD$, $\triangle DOE$ is isosceles. Therefore, $\angle DOE = \alpha$. Since $\angle BDO$ is an exterior angle to $\triangle DOE$, it is equal in measure to the sum of the opposite interior angles of the triangle, i.e. $\angle BDO = 2\alpha$. Now $\triangle BOD$ is isosceles since two of its sides are radii of the circle. Thus $\angle DBO = \angle BDO = 2\alpha$. Since $\angle BOA$ is an exterior angle to $\triangle BOE$, it is equal in measure to the sum of $\angle E = \alpha$ and $\angle EBO = 2\alpha$. Thus $\angle BOA = 3\alpha$, which means that the value k in the problem is $\frac{1}{3}$.

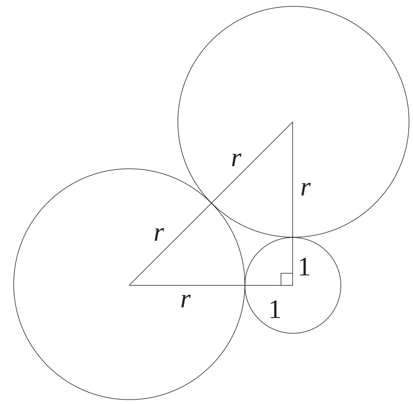
2. Since there are 5 regions of equal area which sum to 180 square units, each region has area 36 square units. The dimensions of the inner square are clearly 6 units on a side, and of the outer square are $\sqrt{180} = 6\sqrt{5}$ units on a side. Let x and y be the dimensions of one of the four congruent regions, where $x < y$. Then $x + y = 6\sqrt{5}$ and $y - x = 6$. On adding these and dividing by 2 we get $y = 3(\sqrt{5} + 1)$, and then it easily follows that $x = 3(\sqrt{5} - 1)$.

3. (a) Note that $5! = 5{\cdot}4{\cdot}3{\cdot}2 = 120$, which ends in the digit 0. Thus $n!$, where $n > 5$, must also end in the digit 0, since $5!$ evenly divides $n!$, for $n > 5$. Thus $n! + 1$ ends in the digit 1 whenever $n \geq 5$, which means that 25 can never evenly divide $n! + 1$ when $n \geq 5$. We are left to examine the cases $n = 4$, 3, 2, and 1. Since $4! + 1 = 24 + 1 = 25$, we see that 25 divides evenly $4! + 1$. Thus $n = 4$ is the largest value of n such that 25 evenly divides $n! + 1$.

   (b) Note first that $(x + 2y) + (y + 2x) = 3x + 3y = 3(x + y)$. Thus three evenly divides the sum of the two numbers. This can be rewritten as $y + 2x = 3(x + y) - (x + 2y)$. Suppose that 3 evenly divides $x + 2y$. This means that $x + 2y = 3k$ for some integer k. Thus $y + 2x = 3(x + y) - 3k = 3(x + y - k)$, which means that 3 evenly divides $y + 2x$.

4. (a) Same as Problem #5(b) on the Junior Paper (Part B).

   (b) Let r be the radius of each of the four larger coins which surround the coin of radius 1. Then by considering two such neighbouring coins and the coin of radius 1 (as in the diagram below) we have a right-angled triangle when we connect the three centres. The Theorem of Pythagoras then implies that



$$\begin{aligned}(2r)^2 &= (r+1)^2 + (r+1)^2\\ 4r^2 &= 2r^2 + 4r + 2\\ 2r^2 - 4r - 2 &= 0\\ r^2 - 2r - 1 &= 0\end{aligned}$$

This quadratic has solutions

$$r = \frac{2 \pm \sqrt{4 - 4(1)(-1)}}{2} = \frac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2}$$

When we reject the negative root, we are left with $r = 1+\sqrt{2}$.

5. (a) The slice is in the shape of an isosceles triangle with two sides equal to the altitude of the equilateral triangular faces of side a and the third side of length a. By using the Theorem of Pythagoras on half of one equilateral triangular face we see that its altitude is given by $a\sqrt{3}/2$. Thus the perimeter of the triangular slice is $a + 2(a\sqrt{3}/2) = a(1+\sqrt{3})$.

   (b) We must now find the area of the triangular slice whose sides we found in part (a) above. Consider the altitude h which splits the isosceles slice into 2 congruent halves. Each half is a right-angled triangle with hypotenuse $a\sqrt{3}/2$ and one side of length $a/2$. The third side is h, which can be found by the Theorem of Pythagoras:

$$\begin{aligned}h^2 &= \left(\frac{a\sqrt{3}}{2}\right)^2 - \left(\frac{a}{2}\right)^2 = \frac{3a^2}{4} - \frac{a^2}{4} = \frac{a^2}{2}\\ h &= \frac{a}{\sqrt{2}} = \frac{a\sqrt{2}}{2}\end{aligned}$$

Thus the area of the slice is

$$\frac{1}{2} \cdot a \cdot \frac{a\sqrt{2}}{2} = \frac{a^2\sqrt{2}}{4}$$