1. Find all the points (x, y) with $y = 3$ that form an isosceles triangle with $A(0, 0)$ and $B(5, 0$.

Solution

Put the points A and B on the plot. Draw the line $y = 3$. Put a point C anywhere on this line and "drag" it along until an isosceles triangle is formed. One such solution is when C lies on the point c_1 where $AC = BC$, i.e. $C_1(2\frac{1}{2}, 3)$. Two other solutions emerge when $AC = AB$. Let $C = (x, 3)$ then by using the Pythagorean theorem $AC^2 = x^2 + 9 = 25 = BC^2$ which gives $x = \pm 4$. So the two points are $C_2(-4, 3)$ and $C_3(4, 3)$. Two more solutions obtained if $AB = BC$. Similarly, by Pythagorean theorem we have $(x - 5)^2 + 9 = 25$ which yields $x = 9$ and $x = 1$. So the two points are $C_4(9, 3)$ and $C_5(1, 3)$. See figure 1.

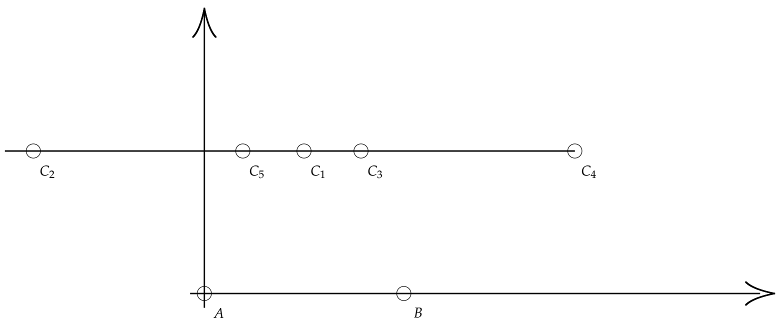


Figure 1: isosceles triangles ABC_i

Answer: $(2\frac{1}{2}, 3), (-4, 3), (4, 3), (9, 3), (1, 3)$

2. Let n be a positive integer for which $n = a^2 + 1902$ and $n = b^2 + 2022$ for some integers a and b. Determine the highest possible value of n.

Solution

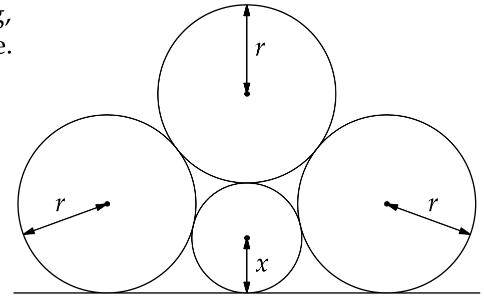
Equate the two expressions to get $a^2 - b^2 = 120$. Hence $(a + b)(a - b) = 120$. Both $(a + b)$ and $(a - b)$ must be integers and $(a + b) > (a - b)$ unless $b = 0$, which is not an option, because then $a = \sqrt{120}$ is not integer. The factors of 120 are $120 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 5$. Make the following table by choosing factors for $a - b$, solving for $a + b$, then solving for a and b using the sum and difference values.

$a - b$	$a + b = 120/(a - b)$	$a = \Big((a - b) + (a + b)\Big)/2$	$b = \Big((a + b) - (a - b)\Big)/2$
2	60	31	29
3	40	not integer	
4	30	17	13
5	24	not integer	
6	20	13	7
8	15	not integer	
10	12	11	1

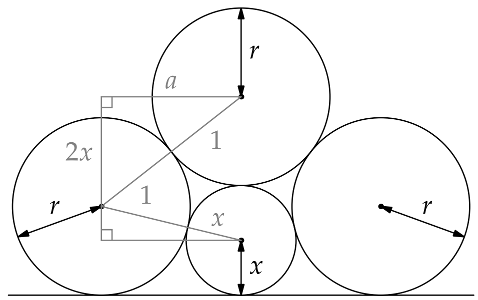
Thus the largest value of a is 31 and thus the largest value of $n = 31^2 + 1902 = 29^2 + 2022 = 2861$.

Answer: 2861

3. In the diagram shown, $r = 1$. All four circles are just touching, and each of the three bottom circles is just touching the line. Calculate x.



Solution



See the diagram above. For the upper right triangle we have

$$a^2 + (2x)^2 = 2^2 \implies a^2 = 4 - 4x^2.$$

For the low right triangle we have

$$a^2 + (1-x)^2 = (1+x)^2 \implies a^2 = (1+x)^2 - (1-x)^2 = 4x$$

Thus

$$a^2 = 4 - 4x^2 = 4x \implies x^2 + x - 1 = 0.$$

This gives a quadratic equation whose only positive root is

$$x = \frac{\sqrt{5}-1}{2}.$$

Answer: $x = \frac{\sqrt{5}-1}{2}$

4. The function f defined by

$$f(x) = \frac{cx}{2x+3},\ x \neq -\frac{3}{2},$$

where c is a real constant, satisfies $f(f(x)) = x$ for all real x except $-\frac{3}{2}$. Determine all possible values of $f(100)$.

Solution

We have

$$f(f(x)) = \frac{cf(x)}{2f(x)+3} = \frac{c\frac{cx}{2x+3}}{2\frac{cx}{2x+3}+3} = \frac{\frac{c^2x}{2x+3}}{\frac{2cx+6x+9}{2x+3}} = \frac{c^2x}{(2c+6)x+9}.$$

Now, $f(f(x) = x$ gives

$$\frac{c^2x}{(2c+6)x+9} = x,$$

which implies $c^2x = (2c+6)x^2 + 9x$, or

$$c^2x = 2cx^2 + 6x^2 + 9x.$$

This is true for every $x \neq -\frac{3}{2}$ only if the coefficients of the corresponding terms be equal on both sides, that is, if $c^2 = 9,\ 2c = -6$, which happens only when $c = -3$. So

$$f(x) = \frac{-3x}{2x+3} \Rightarrow f(100) = \frac{-300}{203}.$$

Answer: $f(100) = \dfrac{-300}{203}$

5. Let P be a permutation of the set {1, 2, 3, 4, 5, 6, 7, 8}. We define a legal operation as moving any number exactly two positions in either direction. For example, if P = {4, 2, 1, 3, 5, 6, 7, 8}, then we can move the number 1 two positions to the left to create {1, 4, 2, 3, 5, 6, 7, 8}, and from this new permutation we can move the number 4 two positions to the right to create {1, 2, 3, 4, 5, 6, 7, 8}. We say that P is solvable if there exists a sequence of legal operations that takes it to the sequence {1, 2, 3, 4, 5, 6, 7, 8}. So from the above example we conclude that {4, 2, 1, 3, 5, 6, 7, 8} is solvable.

   (a) Prove that $P = \{1, 8, 3, 2, 6, 5, 4, 7\}$ is solvable.
   (b) Prove that $P = \{1, 8, 3, 2, 6, 5, 7, 4\}$ is not solvable.
   (c) Let P be a randomly chosen permutation. Determine the probability that P is solvable.

Solution

If it is possible to unscramble a permutation back to the ordered set, the following algorithm will always achieve it. Begin with 1. Move it left as many skips of two as possible without overshooting the desired position. This will either put it in the desired position or one spot to the right thereof. In the latter case take the number sitting to the left of 1 and skip it two spots to the right. Repeat sequentially to put 2, 3, etc. in the desired positions. The set in (a) is unscrambled as follows:

$$\{1,8,3,2,6,5,4,7\} \longrightarrow \{1,2,8,3,6,5,4,7\} \longrightarrow \{1,2,3,6,8,5,4,7\} \longrightarrow \{1,2,3,6,4,8,5,7\} \longrightarrow$$
$$\{1,2,3,4,8,6,5,7\} \longrightarrow \{1,2,3,4,5,8,6,7\} \longrightarrow \{1,2,3,4,5,6,7,8\}\ .$$

Hence the set in (a) is solvable. Repeating the same process for the set in (b) gives

$$\{1,8,3,2,6,5,7,4\} \longrightarrow \{1,2,8,3,6,5,7,4\} \longrightarrow \{1,2,3,6,8,5,7,4\} \longrightarrow \{1,2,3,4,6,8,5,7\} \longrightarrow$$
$$\{1,2,3,4,5,6,8,7\}\ .$$

This ends up in a set that no permissible operation can reduce to the ordered set. Hence the set in (b) is not solvable.

The unscrambling process will always end in either the ordered set or the final set in (b), although the sequence of operations is not unique in its order or number. This allows to refer to all solvable permutations as equivalent to the ordered set and to one another. Similarly all unsolvable ones are equivalent to the final set in (b) and one another. For a set of length n the probability that P is solvable equals the number of solvable permutations divided by $n!$. Suppose the number of solvable permutations is k and thus the number of unsolvable ones is $n! - k$. Observe that swapping the last two numbers in a solvable permutation results in an unsolvable permutation and vice versa. In order to see this suppose the permutation $\{i_1, i_2, \ldots i_{n-1}, i_n\}$ is solvable. There is no legitimate sequence of operations to convert $\{i_1, i_2, \ldots i_n, i_{n-1}\}$ into $\{i_1, i_2, \ldots i_{n-1}, i_n\}$. Thus they are not equivalent. The reverse argument is identical. Thus there is a one-to-one correspondence between solvable and unsolvable permutations. Therefore $n! - k = k$, which gives $k = \frac{1}{2}n!$ and thus the requested probability is $\frac{1}{2}$.

Answer: $\frac{1}{2}$