1. Find all the points (x, y) with $y = 3$ that form an isosceles triangle with $A(0, 0)$ and $B(5, 0$.

Solution

Put the points A and B on the plot. Draw the line $y = 3$. Put a point C anywhere on this line and "drag" it along until an isosceles triangle is formed. One such solution is when C lies on the point c_1 where $AC = BC$, i.e. $C_1(2\frac{1}{2}, 3)$. Two other solutions emerge when $AC = AB$. Let $C = (x, 3)$ then by using the Pythagorean theorem $AC^2 = x^2 + 9 = 25 = BC^2$ which gives $x = \pm 4$. So the two points are $C_2(-4, 3)$ and $C_3(4, 3)$. Two more solutions obtained if $AB = BC$. Similarly, by Pythagorean theorem we have $(x-5)^2 + 9 = 25$ which yields $x = 9$ and $x = 1$. So the two points are $C_4(9, 3)$ and $C_5(1, 3)$. See figure 1.

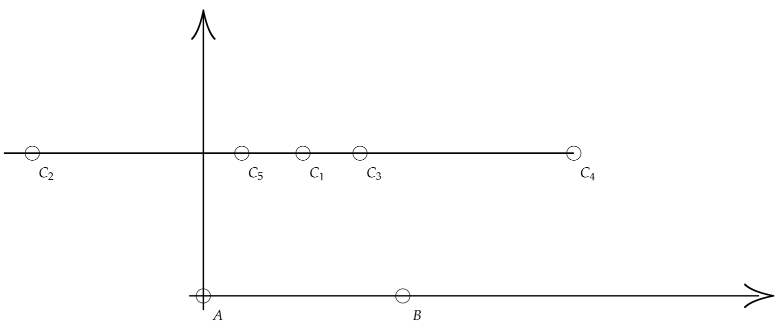


Figure 1: isosceles triangles ABC_i

Answer: $(2\frac{1}{2}, 3), (-4, 3), (4, 3), (9, 3), (1, 3)$

2. In a generous mood, David gave half of his money to Jim. Jim then gave one quarter of the money he then had to David. Each of them ended up with 75 dollars. How much money did each have to start with?

Solution

Suppose David and Jim have initially D and J dollars respectively. The following table shows how their amount of money changes.

	Initial amount	David $\rightarrow$ Jim	Jim $\rightarrow$ David
David	D	$D - \frac{D}{2} = \frac{D}{2}$	$\frac{D}{2} + \frac{1}{4}\left(J + \frac{D}{2}\right)$
Jim	J	$J + \frac{D}{2}$	$J + \frac{D}{2} - \frac{1}{4}\left(J + \frac{D}{2}\right)$

Since they end up with the same amount we must have

$$\frac{D}{2} + \frac{1}{4}\left(J + \frac{D}{2}\right) = J + \frac{D}{2} - \frac{1}{4}\left(J + \frac{D}{2}\right)$$
$$\frac{5D}{8} + \frac{J}{4} = \frac{3J}{4} + \frac{3D}{8}$$
$$\frac{D}{4} = \frac{J}{2}$$
$$D = 2J$$

The total amount of money does not change so if each ends up with 75$, then the total amount must be 150$, and because $D = 2J$, David has started with 100$ and Jim with 50$.

Answer: David: 100$, Jim: 50$

3. Let n be a positive integer for which $n = a^2 + 1902$ and $n = b^2 + 2022$ for some integers a and b. Determine the highest possible value of n.

Solution

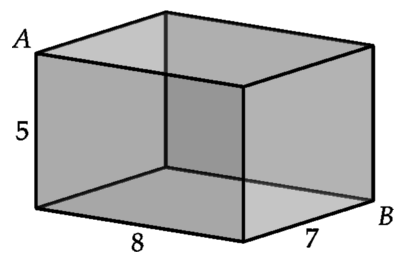
Equate the two expressions to get $a^2 - b^2 = 120$. Hence $(a+b)(a-b) = 120$. Both $(a+b)$ and $(a-b)$ must be integers and $(a+b) > (a-b)$ unless $b = 0$, which is not an option, because then $a = \sqrt{120}$ is not integer. Pairs of positive integers that multiply to 120 are $1 \cdot 120$, $2 \cdot 60$, $3 \cdot 40$, $4 \cdot 30$, $5 \cdot 24$, $6 \cdot 20$, $8 \cdot 15$, and $10 \cdot 12$. Make the following table by choosing factors for $a-b$, solving for $a+b$, then solving for a and b using the sum and difference values.

$a-b$	$a+b = 120/(a-b)$	$a = \big((a-b)+(a+b)\big)/2$	$b = \big((a+b)-(a-b)\big)/2$
2	60	31	29
3	40	not integer	
4	30	17	13
5	24	not integer	
6	20	13	7
8	15	not integer	
10	12	11	1

The largest value of a is 31 and thus the largest value of $n = 31^2 + 1902 = 29^2 + 2022 = 2861$.

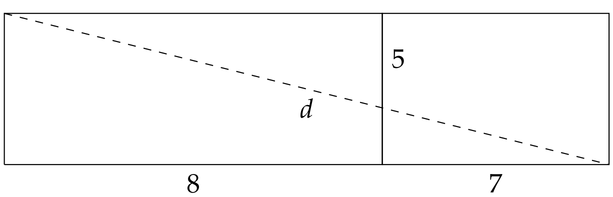
Answer: 2861

4. An ant crawls from corner A to corner B in a $5\,\text{m} \times 7\,\text{m} \times 8\,\text{m}$ room. Calculate the shortest distance it can crawl.

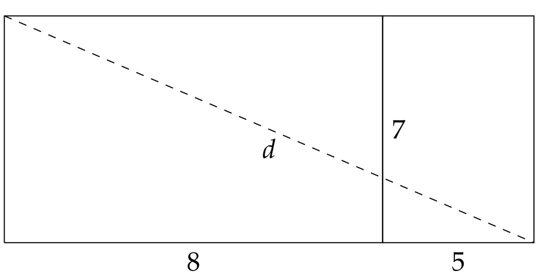


Solution

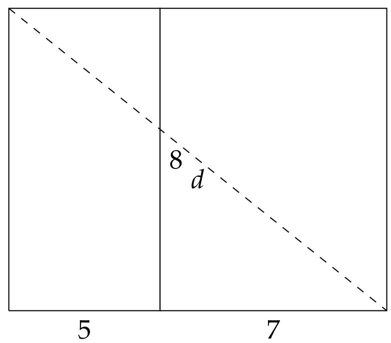
The shortest path will be crossing over two rectangular sides of the box. If we unfold the box, there are three possibilities. We calculate the distance d in each case:



$d = \sqrt{5^2 + 15^2} = \sqrt{250}$ m



$d = \sqrt{7^2 + 13^2} = \sqrt{218}$ m



$$d = \sqrt{8^2 + 12^2} = \sqrt{208} \text{ m}$$

The shortest distance is $d = \sqrt{208}$ m.

Answer: $\sqrt{208}$

5. The function f defined by

$$f(x) = \frac{cx}{2x+3}, \; x \neq -\frac{3}{2},$$

where c is a real constant, satisfies $f(f(x)) = x$ for all real x except $-\frac{3}{2}$. Determine all possible values of $f(100)$.

Solution

We have

$$f(f(x)) = \frac{cf(x)}{2f(x)+3} = \frac{c\frac{cx}{2x+3}}{2\frac{cx}{2x+3}+3} = \frac{\frac{c^2x}{2x+3}}{\frac{2cx+6x+9}{2x+3}} = \frac{c^2x}{(2c+6)x+9}.$$

Now, $f(f(x) = x$ gives

$$\frac{c^2x}{(2c+6)x+9} = x,$$

which implies $c^2x = (2c+6)x^2 + 9x$, or

$$c^2x = 2cx^2 + 6x^2 + 9x.$$

This is true for every $x \neq -\frac{3}{2}$ only if the coefficients of the corresponding terms be equal on both sides, that is, if $c^2 = 9, \; 2c = -6$, which happens only when $c = -3$. So

$$f(x) = \frac{-3x}{2x+3} \;\Rightarrow f(100) = \frac{-300}{203}.$$

Answer: $f(100) = \frac{-300}{203}$

6. Alice, Betty and Carol are competing with each other in a series of math tests. After each test, x points were awarded to the person with the highest score on the test, y points were awarded for the second-highest score, and z points were awarded for the third highest score, with x, y and z being distinct positive integers. Alice scored a total of 20 points, Betty scored a total of 10 points, and Carol scored a total of 9 points.

(a) Prove that there had to be exactly 3 math tests.

(b) If Betty placed first in the Algebra test, who placed second in the Geometry test?

Solution

(a) The total number of points scored is $20 + 10 + 9 = 39$. Let n be the number of tests, then $n(x + y + z) = 39$. Since n, x, y and z are all positive integers, then $x + y + z$ and n must be factors of 39 i.e. 1 and 39 or 3 and 13. On the other hand, we have $x > y > z$ so $x + y + z \geq 6$. Hence the only choice among divisors of 39 is $x + y + z = 13$ and $n = 3$.

(b) Suppose that Betty was 2nd in Geometry. Then Betty has at least a first and a second, so her score of 10 is greater than $x + y$. ($x + y < 10$.) Since x and y are distinct positive integers, this means y is at most 4, which then means that z (positive integer less than y) is at most 3. Then $x + y + z < 10 + 3$, which is false: $x + y + z = 13$.

Now suppose that Alice was 2nd in Geometry. Since she didn't win Algebra, the best that she could have done to achieve a score of 20 is one first place and two second places. This means that $x + 2y$ is greater than or equal to 20 (with equality if she did get one first and two seconds). Meanwhile, the worst that Betty could have done is a first place and two third places, so $x + 2z$ is less than or equal to 10 (with equality if she did get one first and two thirds). Subtracting Betty's score from Alice's score, we find that $2y - 2z$ is greater than or equal to 10, so $y - z$ is greater than or equal to 5. Since y and z are distinct positive integers, this means y is greater than or equal to 6. But then x is greater than or equal to 7, which means $x + y$ is greater than or equal to 13. This is false, since $x + y + z = 13$.

So it must be Carol who was second in Geometry.

We can write a system of equations for x, y and z as follows

Contestant	Total points		Algebra		Geometry		the 3rd test
Alice	20	=	y	+	x	+	x
Betty	10	=	x	+	z	+	z
Carol	9	=	z	+	y	+	y

The solution is $x = 8$, $y = 4$ and $z = 1$.

Answer: (a) number of test = 3, (b) Carol