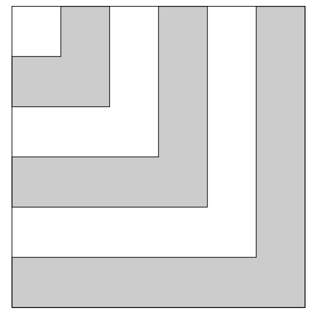
BRITISH COLUMBIA SECONDARY SCHOOL MATHEMATICS CONTEST, 2023

Senior Preliminary Problems & Solutions

1. Two sides of a 6 cm by 6 cm square are divided into equal parts to construct the shaded and unshaded regions shown below. The ratio of shaded to unshaded area is



(A) 3 : 1 (B) 5 : 3 (**C**∗) 7 : 5

(D) 3 : 2 (E) 6 : 5

Solution

The area of shaded region is 21 and the area of unshaded region is 15, so the ration is $\frac{21}{15} = \frac{7}{5}$.

Answer: C

2. Suppose $\dfrac{\frac{1}{x}+\frac{1}{y}}{\frac{1}{x}-\frac{1}{y}} = 2023$. What is $\dfrac{x+y}{x-y}$?

(A) 2023 (**B**∗) -2023 (C) 1 (D) $\frac{1}{2023}$ (E) $-\frac{1}{2023}$

Solution

Multiply top and bottom by xy to get $\frac{y+x}{y-x} = 2023$, so answer is -2023.

Answer: B

3. Mary tells a secret to 6 people. Each of them tells 5 more new people. Each of them tells 4 more people. Each of them tells 3 more, each of whom tell 2 more, each of whom tells one more. At this point, how many people know the secret?

(**A**∗) 1957 (B) 720 (C) 1237 (D) 873 (E) 1593

Solution

Build a tree diagram with Mary at the top, then 6, then $6 \times 5 = 30$, then $6 \times 5 \times 4 = 120$, then 360, then 720, then 720 again. The total is 1957.

Answer: A

4. A pentagon has corners labelled A, B, C, D, E, in clockwise order. A frog starts at corner A and hops from corner to corner clockwise. On Day 1, she makes one hop, finishing the day at B. On Day 2, she makes two hops (B to C and C to D), finishing the day at D. She continues in this way, making three hops clockwise on Day 3, etc. Which corner is she at when Day 2023 ends?

(A) A (**B**∗) B (C) C (D) D (E) E

Solution

Every time five days pass, she's back at A, since in those five days she has hopped $5n+1$, $5n+2$, $5n+3$, $5n+4$, $5n+5$ times with the total of $25n+15$ times, which is a multiple of 5. So her position at the end of Day 2023 is the same as her position at the end of Day 3, which is B – six hops clockwise from A.

Answer: B

5. You have a digital clock that shows hours and minutes, but not seconds. At one point you glance at it, and you see that the time is 1:15. Exactly 40 seconds later you glance at it again, and it says 1:16. Exactly 90 seconds after that, you glance at it again, and it says 1:17. If you glance at it again 20 seconds later, what is the probability that it will say 1:18?

(A) 0 (B) $\frac{1}{3}$ (C) $\frac{1}{2}$ (**D**$*$) $\frac{2}{3}$ (E) 1

Solution

From first glance (G1), clock can be anywhere from start of 1:15 to end of 1:15. From second glance (G2), we know G1 happened between 1:15:20 and 1:16:00, and G2 happened between 1:16:00 and 1:16:40. From G3, we know G2 must have happened between 1:16:00 and 1:16:30, and G3 must have happened in the last 30 seconds of 1:17. Thus exactly 20 seconds later, there is a 2/3 chance that the clock will say 1:18.

Answer: D

6. Suppose $a < b < c < d < e$ are the weights of 5 pumpkins. Weighed two at a time, the weights of the pairs are $3, 5, 6, 8, 9, 11, 12, 13, 15$, and 18. Then c is

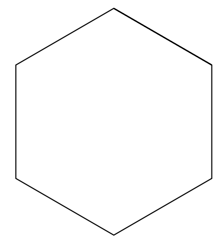
(A) 3.5 (**B**$*$) 4 (C) 4.5 (D) 5 (E) 5.5

Solution

We know that the smallest sum is $a+b=3$, and the largest sum is $d+e=18$. If we add all the sums, we'll have added all the individual weights four times, so $4a+4b+4c+4d+4e=100$ and $a+b+c+d+e=25$. Now $c=25-(a+b+d+e)=25-3-18=4$.

Answer: B

7. Determine the number of ways to position the numbers 1 through 6 on the corners of a hexagon, one number per corner, so that consecutive numbers aren't placed on adjacent corners. For example, the numbers 2 and 4 can't be on corners that connect to the corner with 3 on it. Two ways to position numbers are considered the same if one be rotated to arrive at the other. Mirror images are not considered the same.



(A) 6 (B) 8 (**C**∗) 10

(D) 12 (E) 14

Solution

Might as well imagine hexagon with top and bottom points, with 1 at the top.

- If 2 is at the bottom, then 3 must be adjacent to 1, on either side. Then 4 is on the other side (away from 3), and can be adjacent to either 1 or 2. 5 must be between 3 and 2, and 6 must be next to 4. This gives four ways.
- If 2 is two steps away from 1 (either side) and 3 is opposite from 2, then 4 must be opposite 1 (to avoid 3, and to ensure that 5 and 6 are separated). 5 must be between 1 and 2, and 6 is between 3 and 4. This gives two more ways.
- If 2 and 3 are both two steps away from 1 (two ways to do this), then 4 must go between 1 and 2 (to avoid 3), and there are two ways to finish with 5 and 6. This gives four more ways.

  Thus there are ten ways in total.

Answer: C

8. What is the sum of all the numbers from 1 to 100 that divisible by neither 4 nor 5?

(A) 2400 (B) 2700 (**C**∗) 3000 (D) 3300 (E) 5050

Solution

The sum of all the numbers from 1 to 100 is $101(50) = 5050$. The sum of the numbers $4 + 8 + 12 + ... + 100 = 4(1 + 2 + 3 + ... + 25) = 4(26)25/2 = 1300$. The sum of the numbers $5 + 10 + 15 + ... + 100 = 5(1 + 2 + 3 + ... + 20) = 5(21)20/2 = 1050$. The sum of the numbers $20 + 40 + 60 + 80 + 100 = 300$. Our answer is $5050 - 1300 - 1050 + 300 = 3000$.

Answer: C

9. If $x + y = 1$ and $x^2 + y^2 = 2$, then $x^3 + y^3$ is

(A) 1.5 (B) 2 (**C**∗) 2.5 (D) 4 (E) 8

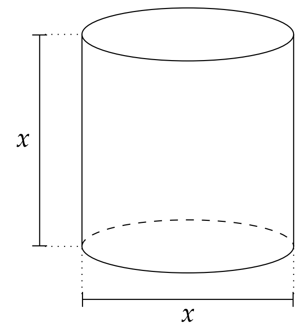
Solution

We have $(x + y)^2 = 1 = x^2 + y^2 + 2xy$, so $2xy = -1$ and $xy = -0.5$. Hence

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2) = 1 \cdot (2 + 0.5) = 2.5$$

Answer: C

10. Let's call a cylinder *standard* if its height equals the diameter of its (circular) base. **See diagram.** We'll say its orientation is *upright* if its top and bottom are its circular ends, and we'll say its orientation is *tipped* if it's lying on its side, so it can roll. What is the volume of the largest tipped standard cylinder that can fit inside an upright standard cylinder of volume 1? (The volume of a cylinder of radius r and height h is $\pi r^2 h$)



(A) $\frac{1}{2}$ (B) $\frac{1}{\sqrt{2}}$ (**C***) $\frac{1}{2\sqrt{2}}$

(D) $\frac{1}{4}$ (E) 1

Solution

Looking down into the large upright cylinder, the tipped cylinder inside will appear as a square inscribed in a circle. The diameter of the square equals the diameter of the circle, so the side length of the square is $\frac{1}{\sqrt{2}}$ times the circle diameter. That means the height and radius of the inner cylinder are $\frac{1}{\sqrt{2}}$ times the height and radius of the outer one, so the volume of the inner cylinder is $\frac{1}{2\sqrt{2}}$.

Answer: C

11. The function $f(x)$ satisfies

$$f(x) = f(x-1) + f(x+1) \qquad \text{for all } x.$$

If $f(1) = 4$ and $f(3) = 1$, then we can find

$$f(2) = f(1) + f(3) = 4 + 1 = 5$$

Find $f(2023)$.

(A) -5 (B) -1 (C) 1 (D) 5 (**E***) 4

Solution

We see $f(x+1) = f(x) - f(x-1)$, so $f(4) = f(3) - f(2) = 1 - 5 = -4$. Similarly, $f(6) = -5 - -4 = -1, f(7) = -1 - -5 = 4$ and $f(8) = 4 - -1 = 5$. The values follow the pattern: $4, 5, 1, -4, -5, -1, 4, 5, 1, -4, -5, -1, ...$ repeating every 6 steps. Since $2023 = 6(337) + 1$, it follows that $f(2023) = f(1) = 4$

Answer: E

12. The number of triples (a, b, c) such that a, b, and c are all positive integers and $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{3}{4}$ is

(A) 10 (B) 16 (C) 31 (**D***) 25 (E) 19

Solution

We notice $(4, 4, 4)$ works. There are 3 shufflings each of $(2, 8, 8)$ and $(3, 3, 12)$ which also work. There are 6 shufflings each of $(2, 6, 12)$, $(2, 5, 20)$ and $(3, 4, 6)$. Notice that as our possible values of a, b and c get large, the unit fractions get too small to making working triples. The solution is$1 + 3 + 3 + 6 + 6 + 6 = 25$.

Answer: D