BRITISH COLUMBIA SECONDARY SCHOOL MATHEMATICS CONTEST, 2023

Senior Final, Part A Problems & Solutions

1. If $2^x = 5$, then $2^{2x+1} =$

(A) 11 (B) 26 (C) 10 (**D**$*$) 50 (E) 64

Solution

If $2^x = 5$, then $2^{2x+1} = 2^x \cdot 2^x \cdot 2^1 = 5 \cdot 5 \cdot 2 = 50$

Answer: D

2. You count the number of printed digits of each page in a book, and count 1035 digits. (For example, there are two printed digits on page 85, and three printed digits on page 126.) How many pages were in the book?

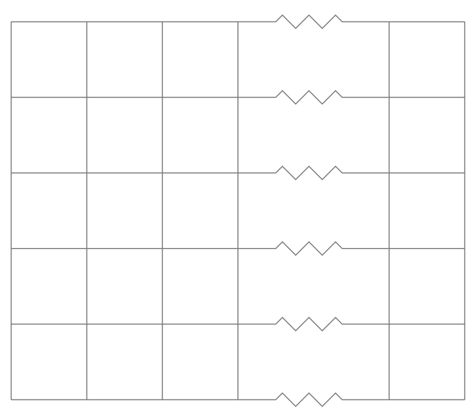
(A) 282 (**B**$*$) 381 (C) 372 (D) 345 (E) 391

Solution

Pages 1–9 give 9 digits. Pages 10–99 give 180 digits. That leaves $1035 - 189 = 846$ pages with three digits. Since $846/3 = 282$, there are 282 pages with three digits, starting with 100. So the book contains 381 pages.

Answer: B

3. How many squares *of any size* are there in the $5 \times n$ diagram for $n \geq 5$?



(A) $20n - 15$

(B) $15n - 15$

(C) $20n - 20$

(**D**$*$) $15n - 20$

(E) Can't be determined

Solution

There are $5 \times n$ 1 by 1 squares. There are $4 \times (n-1)$ 2 by 2 squares. There are $3 \times (n-2)$ 3 by 3 squares. There are $2 \times (n-3)$ 4 by 4 squares. There are $1 \times (n-4)$ 5 by 5 squares. The total is $15n - 20$ squares.

Answer: D

4. Three people, Pat, Nat, and Cat, know each other well. Each either always tells the truth, or always lies. They each make a statement.

- Pat: (Can't be heard.)

- Nat: Pat claimed to be a liar.
- Cat: Nat lied.

From the following statements, choose one that is true.

(A) Pat is telling the truth

(B) Nat is telling the truth

(**C**∗) Cat is telling the truth

(D) They are all lying

(E) They are all telling the truth

Solution

Nat must be lying, since neither a liar nor a truth teller would claim to be a liar. So Cat must be telling the truth.

Answer: C

5. The sum of the first n terms of a sequence is $n(n-1)(2n+3)$. The 12th term of the sequence is?

(**A**∗) 814 (B) 868 (C) 962 (D) 1256 (E) 2650

Solution

The sum of the first twelve terms is the 12th term (the value we want) plus the sum of the first eleven terms. So we'll subtract the sum of the first 11 terms from the sum of the first 12 terms:

$$12 \cdot 11 \cdot 27 - 11 \cdot 10 \cdot 25 = 11 \cdot (12 \cdot 27 - 10 \cdot 25) = 11(324 - 250) = 11 \cdot 74 = 814$$

Answer: A

6. If a is an integer and the tens digit of a^2 is 7, what it the units (ones) digit of a^2?

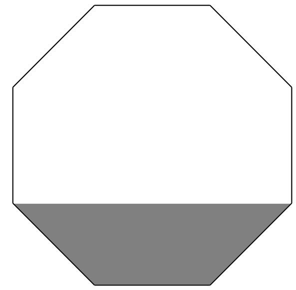
(A) 1 (B) 4 (C) 5 (**D**∗) 6 (E) 9

Solution

Note that $(10x+y)(10x+y) = 100x^2 + 20xy + y^2$. So we need $20xy + y^2$ to end in the digits $7z$. This means that y^2 must have an odd tens digit. So y must be either 4 or 6, both of which square to have 6 in the ones digit. (For example, $20 \cdot 2 \cdot 4 + 4^2 = 160 + 16 = 176$.)

Answer: D

7. A "regular" shape is one where all the sides are of equal length and all angles have the same degree measure. A regular octagon is shown below, with all sides of length 1. What proportion of the total octagon area is captured in the shaded region?



(A) $\frac{1}{2\sqrt{2}}$ (B) $\frac{1}{3}$

(**C**$*$) $\frac{1}{4}$ (D) $\frac{3}{8}$

(E) $\frac{1}{3\sqrt{2}}$

Solution

- The area of the center square is 1,
- four rectangles $4 \cdot 1\frac{1}{\sqrt{2}}$
- four triangles $4 \cdot \frac{1}{2} \cdot \left(\frac{1}{\sqrt{2}}\right)^2$
- total area $2 + \frac{4}{\sqrt{2}} = 2 + 2\sqrt{2}$

So the proportion simplifies to $\frac{1}{4}$.

$\boxed{\textbf{Answer: C}}$

8. A five-digit number is made by randomly arranging the digits of 14874. What is the probability that the number is divisible by 4?

(A) 0.1 (B) 0.2 (C) 0.25 (**D**$*$) 0.3 (E) 0.4

Solution

There are 60 permutations of the digits in 14784.

- 24 of those permutations end in 1 or 7, and are not divisible by 4.
- 12 end in 8, and are good only if the second last digit is one of the 4's. This happens half of the time, giving 6 permutations that are divisible by 4.
- 24 end in 4, and are good only if the second last digit is 4 or 8. This happens half of the time, giving 12 more permutations that are divisible by 4.

So the probability is $\frac{18}{60} = 0.3$.

$\boxed{\textbf{Answer: D}}$

9. Suppose $x + \frac{1}{x} = 5$. Then $x^3 + \frac{1}{x^3} =$

(**A**$*$) 110 (B) 125 (C) 95 (D) 100 (E) 75

Solution

We cube both sides of the equation $x + \frac{1}{x} = 5$ to get

$$x^3 + 3x^2 \cdot \frac{1}{x} + 3x \cdot \left(\frac{1}{x}\right)^2 + \left(\frac{1}{x}\right)^3 = 125,$$

which can be rewritten as

$$x^3 + 3x \cdot \frac{1}{x}\left[x + \frac{1}{x}\right] + \frac{1}{x^3} = 125$$

Since $x + \frac{1}{x} = 5$, this gives

$$x^3 + 15 + \frac{1}{x^3} = 125,$$

so we conclude that

$$x^3 + \frac{1}{x^3} = 110.$$

Answer: A

10. Twins Lucas and Lucia are going to their cousins' house, which is 10km from their home. To save time, they share a bicycle in the following way: Lucas starts out on the bicycle, with Lucia walking behind, then he leaves the bicycle at some point and walks the rest of the way. Lucia walks until she reaches the bicycle, then rides the bicycle the rest of the way. Lucas walks at a rate of 4 km/h while Lucia walks at 5 km/h; Lucas bikes at a rate of 10 km/h while Lucia bikes at 8 km/h. They reach their cousins' house at the same time. How long was the bicycle left without a rider?

    (A) about 15 minutes

    (**B**∗) slightly more than half an hour

    (C) a little less than 45 minutes

    (D) slightly less than 1 hour

    (E) a little more than 1 hour 10 minutes

    Solution

    Let Lucas bike x km out of the 10 km; consequently, Lucia walks x km. Since they take the same time to complete the 10 km-journey, so the time equation yields

$$\frac{x}{10} + \frac{10-x}{4} = \frac{x}{5} + \frac{10-x}{8}$$

    Solving for x, one gets $x = 50/9$. Thus the time difference is $\frac{50/9}{5} - \frac{50/9}{10} = \frac{50/9}{10} = \frac{5}{9}$ of an hour. This is $60 \times \frac{5}{9} = 33\frac{1}{3}$ minutes, or slightly more than half an hour.

Answer: B