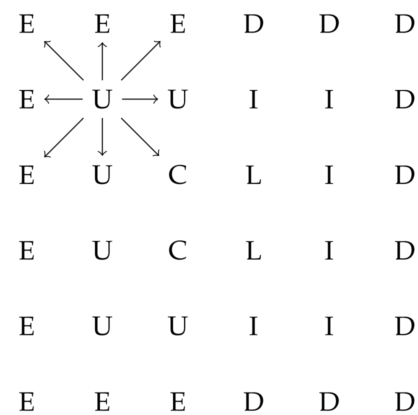
BRITISH COLUMBIA SECONDARY SCHOOL MATHEMATICS CONTEST, 2023

Junior Final, Part B Problems & Solutions

1. The word EUCLID can be spelled by tracing paths through the given array of letters.

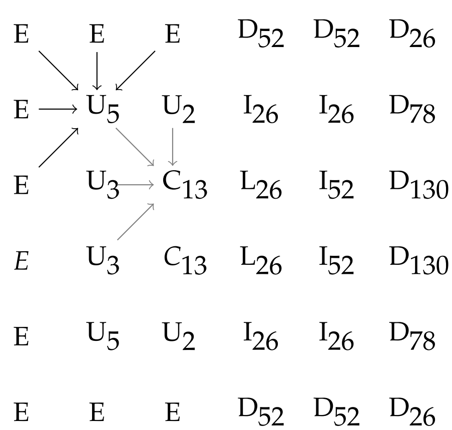
As shown in the diagram, steps to adjacent letters horizontally, vertically, or diagonally are allowed.



Determine the number of different paths which spell the word EUCLID.

Solution

The subscript n of a letter X_n indicates n choices for the previous letter to lead to X. For example, the upper left U_5 shows that five E's lead to that U. The upper C_{13} gets its subscript 13 from adding the subscripts of U that leads to this C, thus $2 + 5 + 3 + 3 = 13$. In short, for the number of paths which spell the word EUCLID, one finds the sum of all subscripts of D, namely, $(52 + 52 + 26 + 78 + 130) \times 2 = 676$ using a horizontal symmetry.



Answer: 676

2. How many integer solutions are there to the equation

$$\frac{P}{Q} - \frac{Q}{P} = \frac{P+Q}{PQ}$$

where $1 \le P \le 9$ and $1 \le Q \le 9$.

Solution

If we multiply the whole equation by PQ, we get $P^2 - Q^2 = P + Q$. Factoring the left side of the equation gives $(P-Q)(P+Q) = P + Q$, and since P and Q are positive, we can divide the equation by $P + Q$ to get $P - Q = 1$ or $P = Q + 1$. The solutions (P, Q) to this equation are $(2,1), (3,2), (4,3), (5,4), (6,5), (7,6), (8,7)$ and $(9,8)$. There are 8 integer solutions.

Answer: 8

3. Find the sum of all distinct four-digit numbers that contain only the digits $1, 2, 3, 4, 5$ each at most once.

Solution

There are 120 distinct four-digit numbers that contain only the digits $1, 2, 3, 4, 5$ each at most once, and each one shows up in $\frac{4}{5}$ of those, or 96, so each digit shows up in each decimal place $\frac{96}{4} = 24$ times. The sum will be $(1 + 2 + 3 + 4 + 5) \cdot 24 \cdot (1000 + 100 + 11 + 1) = 399960$.

Answer: 399960

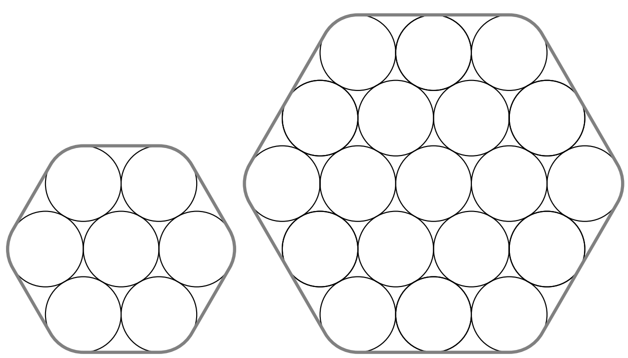
4. Suppose n is an integer greater than 1, that leaves the same non-zero remainder when divided into 1108, 1453, 1844, and 2258. Find n.

Solution

We know $1108 = an + r$, $1453 = bn + r$, $1844 = cn + r$, and $2258 = dn + r$. Therefore, if we subtract any two of our numbers, we should get a multiple of our number n.The differences of our pairs of numbers are 345, 414, 391, and 1150. The greatest common divisor of those numbers is 23.

Answer: 23

5. Circles of radius 1 can be arranged neatly in the shape of a hexagon, if you have an appropriate number of them. Two are shown below. The shape that is formed by an elastic stretched around the outside of one of these will be called a *circagon*, and we refer to the circagon with $6n$ circles touching its elastic as C_n. So C_1 and C_2 are the figures below.



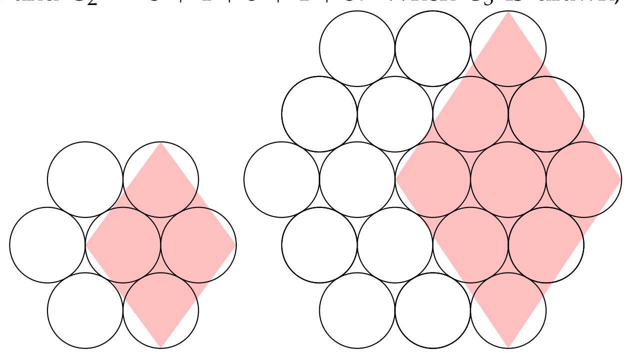
(a) How many circles are used to form C_n?

(b) What is the perimeter of C_n? This is the length of the stretched elastic.

(c) What is the area of C_n? This is the area inside the stretched elastic.

Solution

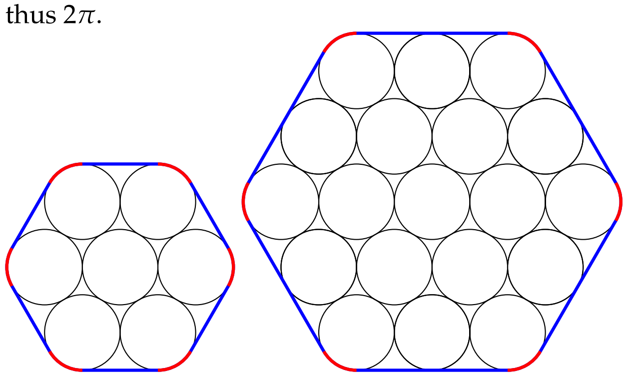
(a) To see the number of circles used to form C_n, one can count circles row by row from the top; for example, $C_1 = 2+3+2$, and $C_2 = 3+4+5+4+3$. When C_3 is drawn, one obtains $C_3 = 4+5+6+7+6+5+4$.



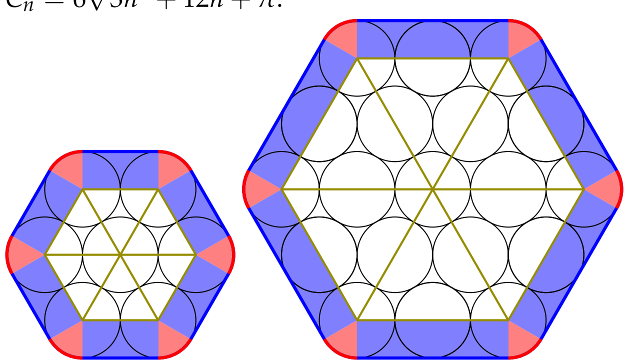
In general,

$$\begin{aligned} C_n &= (n+1)+(n+2)+\cdots+2n+(2n+1)+2n+\cdots+(n+2)+(n+1) \\ &= (2n+1)n+(n+1)^2 \\ &= 3n^2+3n+1 \end{aligned}$$

(b) The perimeter of C_n or the length of the stretched elastic around C_n is $6\times 2n+2\pi$ because each side of C_n contains $n+1$ circles, contributing to n diameters. Since each radius is of length 1, a diameter has length 2. The arc length from six turns total the circumference of a circle of radius 1, thus 2π.



(c) The area of C_n, namely, the area inside the stretched elastic, contains six equilateral triangles of side length $2n$, six $2n\times 1$ rectangles, and one circle of radius 1. Therefore, total area for $C_n = 6\sqrt{3}n^2+12n+\pi$.



Answer: (a) $3n^2+3n+1$**, (b)** $12n+2\pi$**, (c)** $6\sqrt{3}n^2+12n+\pi$**.**