BRITISH COLUMBIA SECONDARY SCHOOL MATHEMATICS CONTEST, 2019

Senior Preliminary Problems & Solutions

1. The number $\dfrac{(2019)^2 + 2019(2020)}{2019 + 2020}$ equals:

   (A) 2020 (B) 2019 (C) 4040 (D) 4038 (E) 4039

   Solution

   $$\frac{(2019)^2 + 2019(2020)}{2019 + 2019} = 2019\left(\frac{2019 + 2020}{2019 + 2020}\right) = 2019$$

   The answer is (B).

2. A group of students charters a bus for \$300. It is agreed that they will share the cost equally. Five students drop out at the last minute, leaving each remaining student to pay \$5 more than originally agreed. The number of students in the original group was:

   (A) 40 (B) 35 (C) 30 (D) 25 (E) 20

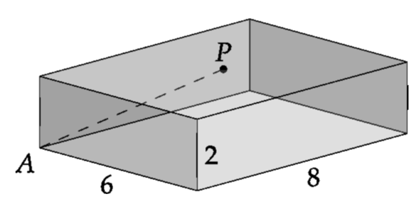
   Solution

   Let x denote the number in the original group. The original cost per student is $\frac{300}{x}$ and the final cost is $\frac{300}{x-5}$.

   $$\begin{aligned} \frac{300}{x} + 5 &= \frac{300}{x-5} \\ 300x - 1500 + 5x^2 - 25x &= 300x \\ 5x^2 - 25x - 1500 &= 0 \\ x^2 - 5x - 300 &= 0 \\ (x-20)(x+15) &= 0 \\ x &= 20 \end{aligned}$$

   The answer is (E).

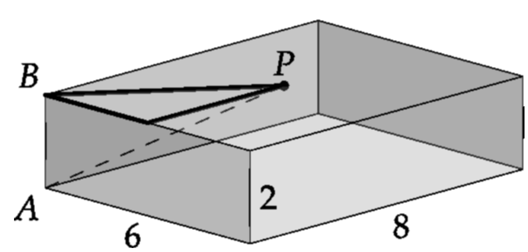
3. A rectangular room measures 2 m by 6 m by 8 m. A cord is fastened to the centre of the ceiling, P, and stretched to reach a lower corner, A. The length of the cord in metres is:



   (A) $\sqrt{24}$ (B) 5 (C) $\sqrt{29}$

   (D) $\sqrt{30}$ (E) none of these

   Solution



Let B represent the corner directly above A. Then BP is the hypotenuse of a right triangle with sides 3 and 4 so it has length $\sqrt{3^2+4^2} = \sqrt{25} = 5$. Now AP is the hypotenuse of the right triangle with sides of length 2 and 5 so it has length $\sqrt{2^2+5^2} = \sqrt{29}$

The answer is (C).

4. Two runners are running a 2-kilometre race on a circular 400-metre track. The ratio of their speeds is 3 : 2. The runners run in opposite directions, beginning at the same time at the start of the track. They both stop when the first person finishes. The number of times they pass each other is:

   (A) 8 (B) 9 (C) 10 (D) 11 (E) 12

Solution

A 2 km race is five laps around the track. The fast runner will finish first and she will pass the slower runner every $3/5$ of the way around the track.

$$\frac{5}{3/5} = \frac{25}{3} = 8\frac{1}{3}$$

They will meet 8 times.

The answer is (A).

5. Let $n > 26$ be an integer. The remainder when $n(n+1)(n+2)$ is divided by $n-2$ is:

   (A) 12 (B) 16 (C) 18 (D) 20 (E) 24

Solution

We have

$$n(n+1)(n+2) = n(n^2+3n+2) = n^3+3n^2+2n.$$

Now use long division:

$$
\begin{array}{rlllll}
 & n^2+ & 5n+ & 12 & \\ \cline{2-5}
n-2 & \big)\, n^3+ & 3n^2+ & 2n & \\
 & n^3- & 2n^2 & & \\ \cline{2-4}
 & & 5n^2+ & 2n & \\
 & & 5n^2- & 10n & \\ \cline{3-5}
 & & & 12n & \\
 & & & 12n- & 24 \\ \cline{4-5}
 & & & & 24
\end{array}
$$

The answer is (E).

Alternative solution

By the remainder theorem $2(2+1)(2+2) = 24$ so the remainder is 24.

6. The perimeter of a right triangle is 14 and its hypotenuse has length 6. The area of the triangle is:

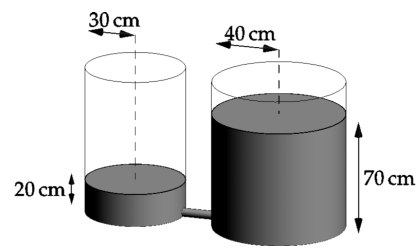
   (A) 6 (B) 7 (C) 8 (D) 9 (E) 14

Solution

$$\begin{aligned}
\text{Area} &= \frac{1}{2}bh \\
b^2+h^2 &= 6^2 = 36 \\
b+h+6 &= 14 \\
b+h &= 8 \\
(b+h)^2 &= 8^2 \\
b^2+2bh+h^2 &= 64 \\
b^2+2bh+h^2-(b^2+h^2) &= 64-36 \\
2bh &= 28 \\
\frac{1}{2}bh &= \frac{28}{4} = 7
\end{aligned}$$

The answer is (B).

7. Two cylindrical tanks, one of radius 40 cm and the other of radius 30 cm, contain oil: the larger one to a depth of 70 cm, and the smaller one to a depth of 20 cm. The bottom of the tanks are at the same level and are connected by a pipe. Oil flows from one tank to the other until the depth in each tank is the same. When the oil stops flowing, the depth, in cm, of the oil in each tank will be:



(A) 52 (B) 50 (C) 48

(D) 45 (E) 43

Solution

The volume of a cylinder is $\pi r^2 h$ so the total volume of oil is

$$\pi(30^2)(20) + \pi(40^2)(70) = 18000\pi + 112000\pi = 130000\pi$$

When both are filled to the same level h:

$$130000\pi = \pi(30^2)h + \pi(40^2)h = 2500\pi h$$

$$h = \frac{130000\pi}{2500\pi} = 52$$

The answer is (A).

8. The last digit of the number $1^{2019} + 2^{2019} + 3^{2019} + 4^{2019}$ is:

(A) 0 (B) 3 (C) 5 (D) 7 (E) 9

Solution

First note that $2019 = 2(1009) + 1 = 4(504) + 3$ and $1^{2019} = 1$.

$2^1 = 2$, $2^2 = 4$, $2^3 = 8$, $2^4 = 16$, $2^5 = 32$ The last digits of 2^x cycle through the four digits 2, 4, 8, 6 so the last digit of 2^{2019} is 8.

Similarly 3^x cycles through the four digits 3, 9, 7, 1 so the last digit of 3^{2019} is 7.

Finally 4^x cycles through the two digits 4 and 6 so the last digit of 4^{2019} is 4.

The final digit of the sum is the final digit of $1+8+7+4 = 20$. The answer is 0.

The answer is (A).

9. A triangle with vertices $A(0,0)$, $B(3,4)$, and $C(2,c)$ has area 5. A possible value of c is:

   (A) -6 (B) $-\frac{2}{3}$ (C) $\frac{2}{3}$ (D) 4 (E) -4

   Solution

   Guess $c = 0$. This gives a triangle with area $\frac{1}{2}(2)(4) = 4$, a little too small. $c = \frac{2}{3}$ and $c = 4$ give smaller triangles and $c = -6$ and $c = -4$ give much larger triangles. The answer must be $c = -\frac{2}{3}$.
   The answer is (B).

   A more satisfying solution:
   The line $x = 2$ intersects the triangle at the points $(2,c)$ and $\left(2, \frac{8}{3}\right)$ dividing it into two smaller triangles, each with base length $|\frac{8}{3} - c|$. The left triangle has height 2 and the right has height 1 so the total area is:

$$\begin{aligned} \frac{1}{2}(2+1)\left|\frac{8}{3} - c\right| &= 5 \\ \left|\frac{8}{3} - c\right| &= \frac{10}{3} \end{aligned}$$

   Case 1:

$$\begin{aligned} \frac{8}{3} - c &= -\frac{10}{3} \\ c &= -\frac{2}{3} \end{aligned}$$

   Case 2:

$$\begin{aligned} \frac{8}{3} - c &= \frac{10}{3} \\ c &= 6 \end{aligned}$$

10. Two hockey teams play to a score of 3, i.e. the game is over as soon as one team gets 3 goals. You make a list showing how the scoreboard changes over time. For example if Team 1 scores, scores again, then Team 2 scores, then Team 1 scores again, then your list is:

$$(0-0), \quad (1-0), \quad (2-0), \quad (2-1), \quad (3-1).$$

   The number of different lists that can be made in this way is:

   (A) 11 (B) 12 (C) 16 (D) 20 (E) 30

   Solution

   The losing team could score 0 goals (1 way), one of the first three goals (3 ways) or two of the first four goals (6 ways).

   Either team could lose so there are $2(1+3+6) = 20$ possible scoring sequences.

   The answer is (D).

11. According to Lewis Carroll in *The Hunting of the Snark* (1876):

   - All Boojums are snarks.

- Every Bandersnatch is a fruminous animal.
- Only animals which frequently breakfast at 5 o'clock tea can be snarks.
- No fruminous animals breakfast at 5 o'clock tea.

Which of the following are true?

I No Boojums are Bandersnatches
II Some snarks can be fruminous animals
III No Bandersnatches breakfast at 5 o'clock tea

(A) only I (B) only II (C) only III
(D) both I and II (E) both I and III

Solution

Every Bandersnatch is a fruminous animal.

Since no fruminous animals breakfast at 5 o'clock tea and only animals which breakfast at 5 o'clock tea can be snarks, no Bandersnatch can be a snark.

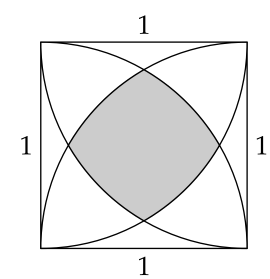
Since all Boojums are snarks, no Boojums are Bandersnatches. Therefore I is true.

Since only animals which frequently breakfast at 5 o'clock tea can be snarks and no fruminous animals breakfast at 5 o'clock tea, no fruminous animal is a snark. Therefore II is false.

Since every Bandersnatch is a fruminous animal and no fruminous animals breakfast at 5 o'clock tea III is true.

The answer is (E).

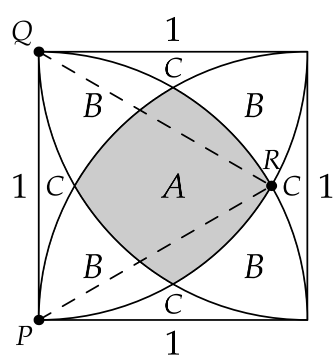
12. The figure shows a unit square with four quarter-circles each having radius 1 and centre at one of the four corners of the square. The area of the shaded region is $a - \sqrt{b} + \frac{\pi}{c}$ where a, b, c are positive integers. The sum $a + b + c$ equals:



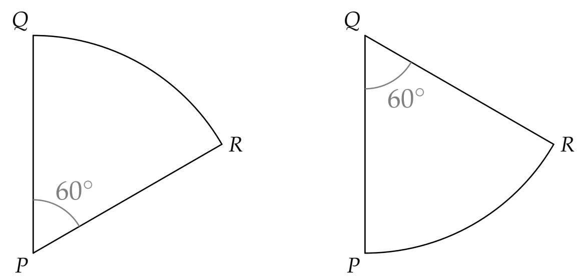
(A) 3 (B) 5 (C) 7
(D) 9 (E) 11

Solution

Denote the area of the shaded region by A, the area of each of the pointy attachments by B and the areas of the 4 remaining cutouts by C. Draw lines from 2 adjacent corners P, Q of the square to the opposite "corner" R of the shaded area, forming an equilateral triangle $\triangle PQR$ of area $\frac{1}{2}(1)(\frac{\sqrt{3}}{2}) = \frac{\sqrt{3}}{4}$:



The circular wedge from Q to R, centered at P, has area $\frac{\pi}{6}$ (since it is $\frac{1}{6}$ of a circle of radius 1). Similarly for the circular wedge from P to R centered at Q:



Adding the areas of these two wedges, we get $A + 2B + C$ plus the area of the equilateral triangle $\triangle PQR$ (which is the intersection of the two wedges). Thus

$$A + 2B + C = 2\frac{\pi}{6} - \frac{\sqrt{3}}{4}.$$

$A + 3B + 2C$ is the area of a quarter circle of radius 1 so

$$A + 3B + 2C = \frac{\pi}{4}.$$

Together these equations imply $B + C = (A + 3B + 2C) - (A + 2B + C) = \frac{\pi}{4} - (2\frac{\pi}{6} - \frac{\sqrt{3}}{4}) = \frac{\sqrt{3}}{4} - \frac{\pi}{12}$.
Finally $A = 1(1) - 4(B + C) = 1 + \frac{\pi}{3} - \sqrt{3}$. Thus $a = 1$, $b = 3$, $c = 3$.
The answer is (C).