BRITISH COLUMBIA SECONDARY SCHOOL MATHEMATICS CONTEST, 2019

Senior Final, Part B Problems & Solutions

1. Place the numbers 1 through 10 in the empty squares in the diagram, so that the sums of the rows and the columns are as indicated. Justify why your answer is unique.

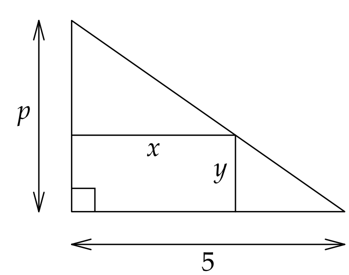
0			0	13
				17
				25
15	6	24	10	

Solution

Row 1 adds up to 13, and column 2 is $1 + 2 + 3 = 6$ (in some order) so Row 1 must be 3, 10. Column 4 adds up to 10, but can't use 1, 2, or 3 (since they're already in column 2) so it's $4 + 6$ or $6 + 4$. Since Row 3 adds up to 25, it can't have 4 (since with 1 or 2 will not be high enough to add to 25 with any other numbers) so Column 4 is 4, 6 (going down). The only possibilities for Column 1 are $7 + 8$ (in some order) and $5 + 9$ (in some order). The only way to make the sums work is to put the higher choices in Row 3 and the lower in Row 2.

0	**3**	**10**	0	13
7	**1**	**5**	**4**	17
8	**2**	**9**	**6**	25
15	6	24	10	

2. A rectangle of width x and height y is inscribed in a right triangle of width 5 and height p, as shown in the diagram. Given that the rectangle has area 6, find the value of x that makes p as small as possible.



Solution

Since ratios of corresponding sides of similar triangles are equal, we have

$$\frac{p-y}{x} = \frac{y}{5-x}.$$

We are given $xy = 6$, so then $y = \frac{6}{x}$ and

$$p = \frac{xy}{5-x} + y = \frac{x(\frac{6}{x})}{5-x} + \frac{6}{x} = \frac{30}{x(5-x)}.$$

Evidently, p is minimized exactly when $z = x(5 - x)$ is maximized. Since the downward parabola represented by z achieves its maximum at the vertex $V = (\frac{5}{2}, \frac{25}{4})$, the value of x that minimizes p is $x = \frac{5}{2}$.

3. (a) Prove that the sum of 3 consecutive odd integers is divisible by 3.

   (b) Prove that the sum of k consecutive odd integers is divisible by k.

   Solution

   (a) Any sum S of three consecutive odd numbers has the form

$$S = (2n-1) + (2n) + (2n+1) = 6n = 3(2n)$$

   for some integer n, and this is evidently a multiple of three.

   (b) More generally, any sum S of k consecutive odd numbers has the form

$$S = \underbrace{(2n+1) + (2n+3) + \cdots + (2n+2k-1)}_{k \text{ terms}} = 2nk + [1 + 3 + 5 + \cdots (2k-1)]$$

   for some integers n and k. By reversing the square-bracketed sum and adding we find

$$\begin{aligned} 2S &= 4nk + [1 + 3 + \cdots (2k-1)] + [(2k-1) + \cdots + 3 + 1] \\ \implies 2S &= 4nk + (2k)k = 4nk + 2k^2, \end{aligned}$$

   so $S = 2nk + k^2 = (2n+k)k$ which is a multiple of k.

4. A bag contains x blue marbles and y red marbles. The numbers x and y are chosen so that if you randomly select two marbles from this bag, there is a 50% chance that the two marbles will be of the same colour.

   (a) If $y = 6$, determine all possible values of x.

   (b) Show that $x + y$ must always be a perfect square.

   Solution

   There are

$$C(x, 2) = \frac{x(x-1)}{2}$$

   ways of selecting two blue balls,

$$C(y, 2) = \frac{y(y-1)}{2}$$

   ways of selecting two red balls, and

$$C(x+y, 2) = \frac{(x+y)(x+y-1)}{2}$$

   ways of selecting any two balls. The probability of selecting two balls of the same colour is then

$$p = \frac{x(x-1) + y(y-1)}{(x+y)(x+y-1)} = \frac{1}{2}$$

$$\begin{aligned} 2x(x-1) + 2y(y-1) &= (x+y)(x+y-1) \\ 2x^2 - 2x + 2y^2 - 2y &= x^2 + xy - x + xy + y^2 - y \\ x^2 - 2xy + y^2 &= x + y \\ (x-y)^2 &= x + y. \end{aligned}$$

   We see $x + y$ is a perfect square, thus answering part (b).

   For part (a), note that if $y = 6$ then $(x-6)^2 = x + 6$ which is a quadratic equation whose roots are $x = 3$ and $x = 10$.

5. We say that a positive integer n is "special" if the first n positive integers can be partitioned into two sets, such that the sum of squares of both sets is equal. For example, $n = 12$ is special because the first 12 integers can be partitioned into sets $\{1, 2, 3, 4, 5, 7, 10, 11\}$ and $\{6, 8, 9, 12\}$, and

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 7^2 + 10^2 + 11^2 = 325 = 6^2 + 8^2 + 9^2 + 12^2.$$

(a) Show that $n = 8$ is special.

(b) Show that $n = 101$ is not special.

(c) Determine whether $n = 102$ is special or not special.

Solution

(a) The sum $1^2 + 2^2 + \cdots + 8^2 = 204$, so each half-sum must be 102. After some systematic fiddling about (consisting mainly of separating the larger squares), we discover that

$$8^2 + 5^2 + 3^2 + 2^2 = 102 = 7^2 + 6^2 + 4^2 + 1^2,$$

so 8 is special.

(b), (c) The above method of solution is hopelessly inefficient for large values of n such as 101 and 102, so we turn to other considerations. We prove 101 is not special by using the fact that if the sum of all the squares is odd, then we will not be able to divide it into two equal parts that are whole numbers. We notice that if there are an odd number of odd squares being added to any number of even squares, the sum will be odd and the original number in question will not be special. Observe further that if the original value is of the form $4k$ or $4k + 3$, there will be an even number of odds, so the sum is even and the value could be special. On the other hand, values of the form $4k + 1$ or $4k + 2$ will include an odd number of odds, so the sum will be odd and the value can not be special. Since 101 is of the form $4k + 1$ and 102 is of the form $4k + 2$, neither can be special.