BRITISH COLUMBIA SECONDARY SCHOOL MATHEMATICS CONTEST, 2019

Senior Final, Part A Problems & Solutions

1. A statistician found the average of 43 numbers to be x. Then, by accident, she included that average x in the data set, and found the average of the 44 numbers to be y. The ratio of y to x is:

   (A) $\frac{43}{44}$ (B) $\frac{44}{43}$ (C) $\frac{45}{44}$ (D) $\frac{44}{45}$ (E) 1

   Solution

   Let S be the sum of the original 43 numbers. Then

   $$\frac{S}{43} = x \implies S = 43x.$$

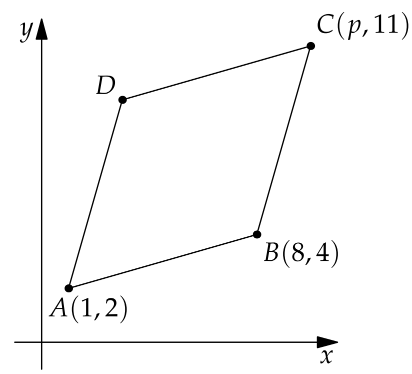
   When the value x is included with the original numbers, the sum of all 44 numbers is $S + x$, so

   $$\frac{S+x}{44} = y \implies S + x = 44y.$$

   Subtracting these two equations gives $y = x$. The correct answer is (E).

2. The quadrilateral $ABCD$ shown has four sides of equal length. The value of p is:

   (A) 9 (B) 10 (C) 11
   (D) 12 (E) 15



   Solution

   The distance from A to B is the same as the distance from B to C. Using the distance formula:

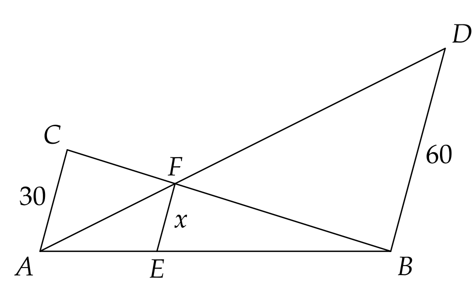
   $$\sqrt{(8-1)^2 + (4-2)^2} = \sqrt{53} = \sqrt{(p-8)^2 + (11-4)^2}.$$

   $$\begin{aligned} 53 &= (p-8)^2 + 49 \\ (p-8)^2 &= 4 \\ p - 8 &= 2 \\ p &= 10. \end{aligned}$$

   The correct answer is (B).

3. If AC, BD, and EF are parallel, then the value of x is:

   (A) 10 (B) 15 (C) 20
   (D) 30 (E) 45



Solution

Using similar triangles ΔEBF and ΔABC, we obtain

$$\frac{EB}{x} = \frac{AB}{30}, \qquad \text{so} \quad EB = \frac{ABx}{30}$$

Also using similar triangles ΔAEF and ΔABD, we obtain

$$\frac{AE}{x} = \frac{AB}{60}, \qquad \text{so} \quad AE = \frac{ABx}{60}$$

Hence, $AB = AE + EB = 3\frac{ABx}{60}$ and thus $x = 20$. The correct answer is (C).

Alternate Solution

We have $\frac{BD}{x} = \frac{AB}{AE}$ and $\frac{AC}{x} = \frac{AB}{BE}$. Since $AE = AB - BE$, we conclude $\frac{BD}{x} = \frac{AB}{AB - BE}$ and hence $\frac{x}{BD} = \frac{AB - BE}{AB}$ which implies $\frac{x}{60} = 1 - \frac{BE}{AB}$ and therefore $\frac{x}{60} = 1 - \frac{x}{30}$. From here we conclude that $x = 20$.

4. The numbers $1, 2, 3, \ldots$ are arranged in the "spiral square" pattern shown. The numbers in the row $1, 4, 15, \ldots$ are circled. The sixth circled number in this sequence is:

(A) 90 (B) 92 (C) 94 (D) 96 (E) 98

$$\begin{array}{ccccccc}
10 & \rightarrow & 11 & \rightarrow & 12 & \rightarrow & 13 \\
\uparrow & & & & & & \downarrow \\
9 & & 2 & \rightarrow & 3 & & 14 \\
\uparrow & & \uparrow & & \downarrow & & \downarrow \\
8 & & \textcircled{1} & & \textcircled{4} & & \textcircled{15} \\
\uparrow & & & & \downarrow & & \downarrow \\
7 & \leftarrow & 6 & \leftarrow & 5 & & 16 \\
 & & & & & & \downarrow \\
 & \cdots & \text{etc.} & \leftarrow & 18 & \leftarrow & 17
\end{array}$$

Solution

We notice that after 2 steps, the pattern is that at each step you add what you added last time plus 8 more. That is,

$$\begin{aligned}
&1 + 3 = 4 \\
&4 + (\underbrace{3 + \mathbf{8}}_{11}) = 15 \\
&15 + (\underbrace{11 + \mathbf{8}}_{19}) = 34 \\
&34 + (\underbrace{19 + \mathbf{8}}_{27}) = 61 \\
&61 + (\underbrace{27 + \mathbf{8}}_{35}) = 96.
\end{aligned}$$

The correct is (D).

5. The equation $2x^2 + 5xy - 12y^2 = 28$ has exactly one solution in positive integers x and y. The sum of x and y is

(A) 11 (B) 12 (C) 13 (D) 14 (E) 15

Solution

We can factor the left side as $(2x - 3y)(x + 4y) = 28$. Let $2x - 3y = a$ and $x + 4y = b$. Then $x = \frac{4a + 3b}{11}$ and $y = \frac{-a + 2b}{11}$ and hence $x + y = \frac{3a + 5b}{11}$. Note that $\frac{3a + 5b}{11}$ should be an integer. All possibilities for $ab = 28$ are $(a,b) = (1,28), (a,b) = (2,14), (a,b) = (4,7), (a,b) = (7,4), (a,b) = (14,2)$, and $(a,b) = (28,1)$. The only time that $\dfrac{3a+5b}{11}$ becomes an integer is when $a = 1$ and $b = 28$, which implies $x + y = \dfrac{3(1)+5(28)}{11} = 13$. The correct answer is (C).

Alternate Solution

Let $x + y = n$. Then $y = n - x$ and hence

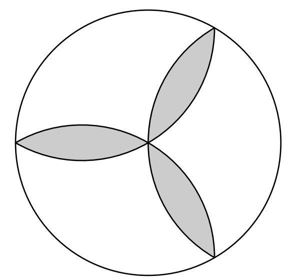
$$2x^2 + 5x(n-x) - 12(n-x)^2 = 28$$

which simplifies to

$$15x^2 - 29nx + 12n^2 + 28 = 0.$$

Using the quadratic equation, we see that $x = \dfrac{29n \pm \sqrt{121n^2 - 1680}}{30}$ and the radical is a perfect square only for $n = 13$.

6. The radius of the circle shown is r, and the radius of each of the arcs is r. The area of the shaded region is:

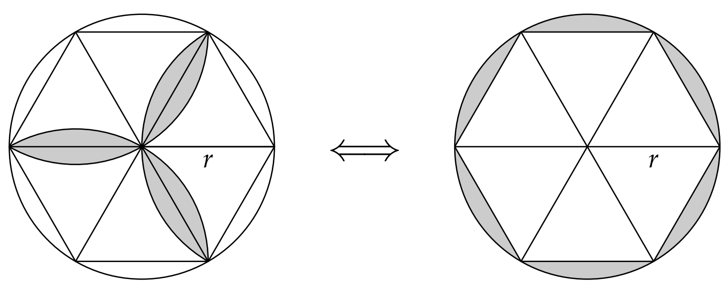


(A) $3\pi r^2$ (B) $\dfrac{3\pi r^2}{2}$ (C) $\dfrac{3\sqrt{3}r^2}{2}$

(D) $\pi r^2 - \dfrac{3\sqrt{3}r^2}{2}$ (E) $(3\sqrt{3} - \pi)r^2$

Solution

This problem is equivalent to finding the area of a circle of radius r minus an inscribed hexagon of diagonal $2r$:



Since a regular hexagon of side length r consists of 6 equilateral triangles (of area $\frac{1}{2}r(\frac{\sqrt{3}}{2}r)$), the answer is

$$\pi r^2 - 6\left(\frac{1}{2}r\left(\frac{\sqrt{3}}{2}r\right)\right) = \pi r^2 - \frac{3\sqrt{3}}{2}r^2.$$

The correct answer is (D).

7. A detective questions four suspects about a crime. He takes the following statements:

Allistair: "Boris or Carmen did it."
Boris: "Allistair or Davina did it."
Carmen: "I did it."
Davina: "I didn't do it."

The 100% accurate lie-detector test indicates that three of the suspects are lying, and one of them is telling the truth. Unfortunately, the results are scrambled and it is impossible to tell which suspect is telling the truth. The crime was committed by:

(A) Allistair (B) Boris (C) Carmen (D) Davina (E) can't be determined

Solution

Suppose Davina told the truth. In this case, Davina didn't do it and since Carmen lied, Carmen didn't do it either. So according to Boris (who lied) neither Allistair nor Davina did it. This leaves Boris as the suspect. But, this means Allistair told the truth, which is not possible. So Davina lied, which means Davina committed the crime. The correct answer is (D).

8. A box contains 10 balls numbered 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10. If three balls are randomly taken out of the box, the probability that the number on one of these balls will be the average of the other two balls is:

(A) $\frac{1}{3}$ (B) $\frac{1}{4}$ (C) $\frac{1}{5}$ (D) $\frac{2}{15}$ (E) $\frac{1}{6}$

Solution

Suppose x, y, z are the numbers on the balls that are taken out, and that (without loss of generality) z is the average of x and y, i.e.

$$z = \frac{x+y}{2}.$$

Then x, y are either both even or both odd (since their sum is divisible by 2).

The number N of triples (x, y, z) in which x, y are either both even or both odd is equal to the number of ways to pick two odd numbers ($C(5,2) = 10$) plus the number of ways to pick two even numbers ($C(5,2) = 10$), i.e. $N = 10 + 10 = 20$. (In each such triple the value of z is then determined by $z = (x+y)/2$). Since there are $C(10,3) = 120$ possible triples, the probability is $20/120 = 1/6$.

The correct answer is (E).

Alternate solution

The total number of choices is 120. The following are the only possible cases that one of the numbers is the average of the other two:

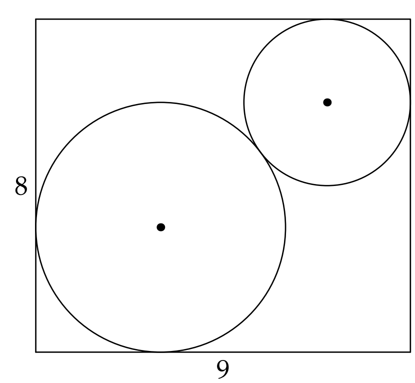
$$\{1,2,3\}, \{1,3,5\}, \{1,4,7\}, \{1,5,9\}, \{2,3,4\}, \{2,4,6\}, \{2,5,8\}, \{2,6,10\},$$
$$\{3,4,5\}, \{3,5,7\}, \{3,6,9\}, \{4,5,6\}, \{4,6,8\}, \{4,7,10\}, \{5,6,7\}, \{5,7,9\},$$
$$\{6,7,8\}, \{6,8,10\}, \{7,8,9\}, \{8,9,10\}.$$

Since there are 20 cases, the probability is $\frac{20}{120} = \frac{1}{6}$.

9. Two circles are inscribed in an 8×9 rectangle as shown. The sum of the radii of the circles is:

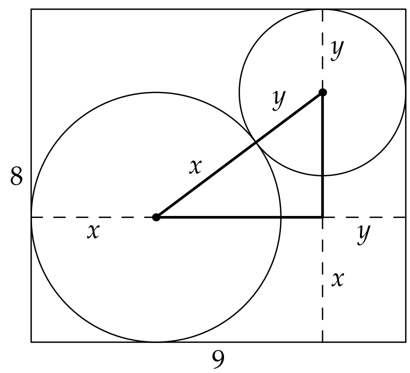
(A) $\frac{9}{2}$ (B) 5 (C) $\sqrt{26}$

(D) $\sqrt{34}$ (E) 6



Solution

Join the centers of the two circles, and this turns out to be the hypotenuse of a right-angled triangle:

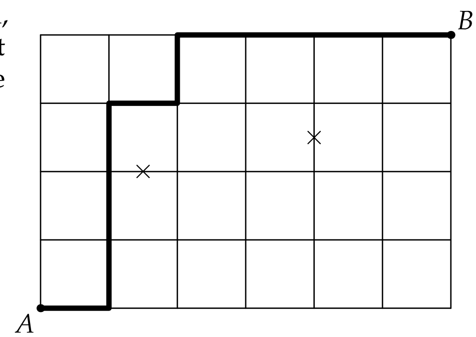


Letting the radii be x and y, we get a right-angled triangle with legs $9-(x+y)$, $8-(x+y)$ and hypotenuse $x+y$. Let $x+y=t$. By Pythagoras theorem we have

$$(9-t)^2+(8-t)^2=t^2 \Rightarrow t^2-34t+145=0 \Rightarrow (t-5)(t-29)=0.$$

The only acceptable answer is $t=5$. The correct answer is (B).

10. You plan to travel from point A to point B on the grid shown, moving only up or to the right along the grid lines, and without crossing an $\times$ mark. One possible path is shown in bold. The number of possible paths is:

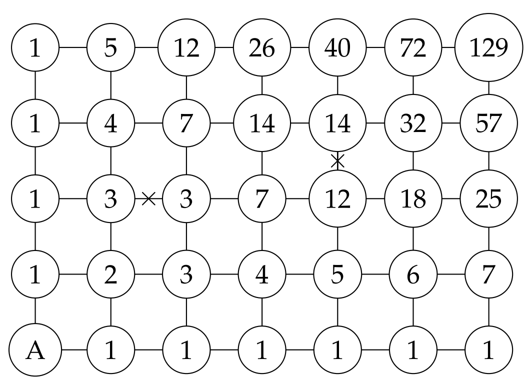


(A) 129 (B) 53 (C) 210

(D) 88 (E) 117

Solution

Starting on the bottom left corner and going right and up, we can count the number of possible paths (similar to the Pascal Triangle). The total number of paths will be 129.



The correct answer is (A).