BRITISH COLUMBIA SECONDARY SCHOOL MATHEMATICS CONTEST, 2019

Junior Final, Part B Problems & Solutions

1. There are 20 people in a math class. Each person is in either grade 8 or grade 9. The average grade of the grade 8 students is 80% and the average grade of the grade 9 students is 90%. If the average grade of all the students is 84%, determine the number of grade 9 students in the class.

   Solution

   Let n be the number of grade 9 students. Then there are $20 - n$ grade 8 students. The sum of grade 8 students' scores (each out of 100) is therefore $80 \times (20 - n)$ and the sum of grade 9 students' scores is $90 \times n$. The sum of all 20 students' scores is $84 \times 20 = 1680$. Therefore,

   $$80(20 - n) + 90n = 1680 \implies 1600 - 80n + 90n = 1680 \implies 10n = 80.$$

   Thus $n = 8$.

2. Place the numbers 1 through 10 in the empty squares in the diagram, so that the sums of the rows and the columns are as indicated. Justify why your answer is unique.

0			0	13
				17
				25
15	6	24	10	

   Solution

   Row 1 adds up to 13, and column 2 is $1 + 2 + 3 = 6$ (in some order) so Row 1 must be 3, 10. Column 4 adds up to 10, but can't use 1, 2, or 3 (since they're already in column 2) so it's $4 + 6$ or $6 + 4$. Since Row 3 adds up to 25, it can't have 4 (since with 1 or 2 will not be high enough to add to 25 with any other numbers) so Column 4 is 4, 6 (going down). The only possibilities for Column 1 are $7 + 8$ (in some order) and $5 + 9$ (in some order). The only way to make the sums work is to put the higher choices in Row 3 and the lower in Row 2.

0	**3**	**10**	0	13
7	**1**	**5**	**4**	17
8	**2**	**9**	**6**	25
15	6	24	10	

3. Each of Xaviera, Yolanda and Zeke picks an integer greater than or equal to 1 and less than or equal to 9. Xaviera multiplies her number by 12, and then adds Yolanda's number. They multiply the result by 10. To this number they add their three original choices. The result is 878. What must have been their chosen integers?

Solution

Let their chosen numbers be x, y and z (with the obvious alphabetical assignments). The given instructions produce the equation

$$10(12x + y) + (x + y + z) = 878$$
$$\implies 121x + 11y + z = 878.$$

Since x, y, z are whole numbers between 1 and 9, x must be as large as possible with $121x < 878$. This gives $x = 7$, in which case the equation above gives

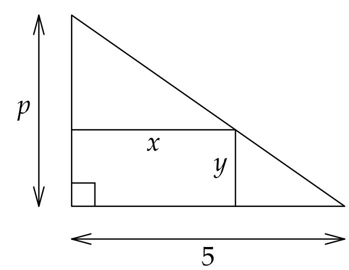
$$121(7) + 11y + z = 878 \implies 11y + z = 31.$$

Again, y must be as large as possible with $11y < 31$. This gives $y = 2$, and so

$$11(2) + z = 31 \implies z = 9.$$

Thus $x = 7$, $y = 2$, and $z = 9$.

4. A rectangle of width x and height y is inscribed in a right triangle of width 5 and height p, as shown in the diagram. Given that the rectangle has area 6, find the value of x that makes p as small as possible.



Solution

Since ratios of corresponding sides of similar triangles are equal, we have

$$\frac{p-y}{x} = \frac{y}{5-x}.$$

We are given $xy = 6$, so then $y = \frac{6}{x}$ and

$$p = \frac{xy}{5-x} + y = \frac{6}{5-x} + y = \frac{6}{5-x} + \frac{6}{x} = \frac{30}{x(5-x)}.$$

Evidently, p is minimized exactly when $x(5-x)$ is maximized. Since this maximum occurs at the vertex of the parabola $z = x(5-x)$ which is halfway between the x-intercepts $x = 0$ and $x = 5$, the value of x that minimizes p is $x = \frac{5}{2}$.

5. For any given integer n, define $n! = 1 \times 2 \times 3 \times \cdots \times n$. For example, $5! = 1 \times 2 \times 3 \times 4 \times 5 = 120$. Determine the smallest positive integer n for which $n!$ is divisible by 10^{100}.

Solution

It suffices to find the smallest n such that there are 100 factors of 5 in the product

$$n! = 1 \times 2 \times \cdots \times n,$$

because then there will also be at least 100 factors of 2, and hence 100 factors of $2 \times 5 = 10$.

The terms 5, 10, 15, 20, 25 contribute $5 + 1 = 6$ factors of 5 (since 25 has an additional factor of 5). This means 25! is divisible by 10^6 (but not 10^7).

From this, we can reason that the terms up to $n = 5 \times 25 = 125$ contribute $5 \times 6 + 1 = 31$ factors of 5 (since 125 has an additional factor of 5) so 125! is divisible by 10^{31}.

The terms up to $n = 3 \times 125 = 375$ contribute $3 \times 31 = 93$ factors of 5, so 375! is divisible by 10^{93}.

We need only 7 more factors of 5 before $n!$ is divisible by 10^{100}. These factors come from the terms $380, 385, 390, 395, 400, 405$. Thus $n = 405$.

Alternate solution

As above, we need that $n!$ have exactly 100 factors of 5.

Notice that if $n = 5k$ then $n!$ contains *at least* k factors of 5. Thus $(5 \times 100)! = 500!$ is certainly divisible by 10^{100}.

But $n = 500$ is not the *smallest* such n, because 500! contains terms like $25, 50, 75, \ldots$ each of which contributes an extra factor of 5 (since these terms are divisible by $25 = 5^2$). Similarly, the terms $125, 250, 375, \ldots$ each contributes a further factor of 5 (since these terms are divisible by $125 = 5^3$).

Thus the number of factors of 5 in $n!$ is

$$\underbrace{\left\lfloor \frac{n}{5} \right\rfloor}_{\#\text{ multiples of } 5} + \underbrace{\left\lfloor \frac{n}{25} \right\rfloor}_{\#\text{ multiples of } 5^2} + \underbrace{\left\lfloor \frac{n}{125} \right\rfloor}_{\#\text{ multiples of } 5^3} + \cdots$$

where $\lfloor x \rfloor$ denotes the value of x rounded down to the nearest whole number.

The number of factors of 5 in 400! is

$$\left\lfloor \frac{400}{5} \right\rfloor + \left\lfloor \frac{400}{25} \right\rfloor + \left\lfloor \frac{400}{125} \right\rfloor = 80 + 16 + 3 = 99$$

so 400! is divisible by 10^{99} but not 10^{100}. We need that $n!$ contain only one more factor of 5. Thus $n = 405$:

$$\left\lfloor \frac{405}{5} \right\rfloor + \left\lfloor \frac{400}{25} \right\rfloor + \left\lfloor \frac{400}{125} \right\rfloor = 81 + 16 + 3 = 100.$$

The smallest positive integer n for which $n!$ is divisible by 10^{100} is $n = 405$.