BRITISH COLUMBIA SECONDARY SCHOOL MATHEMATICS CONTEST, 2018

Senior Final, Part B Problems & Solutions

1. A bag contains 4 red, 5 blue, and 6 green marbles. Maelle is blindfolded and asked to take a handful of marbles from the bag. Find (with explanation) the smallest number she must take to be certain that her handful contains:

   (a) At least one marble of each colour.
   (b) At least two marbles of the same colour.
   (c) At least two marbles of one colour, and at least two marbles of another colour.
   (d) At least two blue marbles and at least two red marbles.

   Solution

   We look at the "worst case" scenarios and suppose someone is selecting Maelle's marbles to make it as difficult as possible to get the right colours.

   a) If she selects all the blue and green marbles first, then she'll have eleven marbles without a red marble. Selecting an additional marble will give marbles of all colours. Thus the minimum is 12.

   b) If the first three are all different, then she won't have two marbles of the same colour. Selecting an additional marble will result in two marbles of the same colour. Thus the minimum is 4.

   c) If picks all six green marbles, and one blue, and one red, then she'll have only one pair of marbles of same colour (green). Picking an additional marble will result either in a blue pair, or a red pair. Thus the minimum is 9.

   d) If she picks all the blue marbles and all the green marbles, and one red marble, she will have 12 marbles, but no red pair. Picking an additional marble (which is red) results in a red pair. Thus the minimum is 13.

2. The sequence $17, 8, 1, A, B, C, D, E, F, G, H, I, J, K, 7, 9, 16$ consists of each integer from 1 to 17, each integer used exactly once. Moreover, the sum of each pair of consecutive terms is a perfect square. (For example, $17 + 8 = 25 = 5^2$, $8 + 1 = 9 = 3^2$, and $1 + A$ are all perfect squares.) Find, with explanation, F, the middle number in this sequence.

   Solution

   We observe that $K + 7 = x^2$. Thus $K = x^2 - 7$, and the possibilities are $x = 3, K = 2$, or $x = 4, K = 9$. Note $K \neq 9$, since 9 already appears in the sequence. Thus $K = 2$ Consequently, $J = x^2 - 2$, and thus $x = 4, J = 14$, is the only choice $J \neq 2, 7, 23$. Now $I = x^2 - 14$, and hence $x = 4, I = 2$, or $x = 5, I = 11$. Since $I \neq 2$, it follows that $I = 11$. Continuing, we have $H = x^2 - 11$. Thus $x = 4, H = 5$ or $x = 5, H = 14$. But $H \neq 14$ (since $J = 14$) and hence $H = 5$. Now $G = x^2 - 5$ and thus $x = 3, G = 4$ or $x = 4, G = 11$ are the only choices since $G \neq 9, 20$. Suppose$G = 11$, then $F = 5, 14$. Since $F \neq 5, 14$ (because $H = 5$ and $J = 14$), it follows $G = 4$. Then $F = 5, 12$ are the only choices. But $F \neq 5$, and thus $F = 12$. Thus $F = 12$.

   Answer: $F = 12$.

3. Find all possible sequences of consecutive positive integers that sum to 100, and explain why your list is complete.

   Solution

   There are two possible sequences: $18 + 19 + 20 + 21 + 22 = 100$ and $9 + 10 + 11 + 12 + 13 + 14 + 15 + 16 = 100$. To see this, let N be the number of integers in the sum and let $x, x + 1, \ldots, x + N - 1$ be the values in the sum. Let x_{av} denote the average value. Then $N \cdot x_{av} = 100$. Suppose N is odd. Then

$x_{av} = x + \frac{N-1}{2}$. So $N \cdot (x + \frac{N-1}{2}) = 100$ and we see that N divides 100. Clearly $N \neq 1, 25$ and therefore $N = 5$. So $x + (\frac{5-1}{2}) = 20$, and hence $x = 18$. Therefore, when N is odd, we get the first sequence above. Suppose N is even. Then $x_{av} = \frac{x+x+(N-1)}{2} = \frac{2x+N-1}{2}$. Thus $N \cdot (2x + N - 1) = 200 = 2^3 \cdot 5^2$. Clearly, $N < 15$. Thus if 5 divides N, then $N = 2 \cdot 5 = 10$. We would then have $2x = 20 - 9 = 11$, which has no solution for x. Thus 5 does not divide N. Given that $2x + N - 1$ is an odd factor of 200, it follows that $2x + N - 1 = 25$ and hence $N = 8$. Consequently, $2x = 25 - 7 = 18$ and $x = 9$. This gives the second sequence.

4. Suppose we have the following array of numbers where in the nth row, the numbers $1, 2, \ldots, n$ occur in the even positions, and the numbers $n, n + 1, \ldots, 2n - 1$ occur in the odd positions as illustrated below.

Row 1:	1 1
Row 2:	2 1 3 2
Row 3:	3 1 4 2 5 3
Row 4:	4 1 5 2 6 3 7 4
Row 5:	5 1 6 2 7 3 8 4 9 5

(a) Find the 50th number in Row 100.
(b) Let $f(n)$ the nth term in row n. For example: $f(1) = 1,\ f(2) = 1,\ f(3) = 4$, and $f(4) = 2$. Determine all n for which $f(n) = 2018$.

Solution

a) The even-numbered entries in row 100 are $1, 2, 3, 4, 5, \ldots$. The 50th entry is the 25th even-numbered entry, so the answer is 25.

b) If n is even, then the nth term in row n is $\frac{n}{2}$. Thus $\frac{n}{2} = 2018 \Rightarrow n = 4036$. If n is odd, then the nth term in row n is $n + \frac{n-1}{2} = \frac{3n-1}{2}$, or $\frac{3n-1}{2} = 2018 \Rightarrow 3n - 1 = 4036$, which does not give an integer solution. Therefore, $n = 4036$ is the only solution.

5. The parabola $y = ax^2 + bx + c$ has vertex at (t, t) and passes through $(-t, -t)$.

(a) If $t = 2$ determine a, b, c.
(b) If $a^2 + b^2 + c^2 = \frac{33}{16}$ determine all possible values of t.
(c) Determine the value of t for which $a^2 + b^2 + c^2$ is minimized.

Solution

a) If $t = 2$, then because $(2, 2)$ and $(-2, 2)$ are on the parabola, we know $4a + 2b + c = 2$ and $4a - 2b + c = -2$. By adding these equations and solving, we get $8a + 2c = 0$, so $c = -4a$. We also know $(2, 2)$ is the vertex, so $-\frac{b}{2a} = 2$, and $b = -4a$ ($= c$ from before). Plugging $(2, 2)$ into the equation $y = ax^2 - 4ax - 4a$, we obtain $4a - 8a - 4a = 2$, so $a = -\frac{1}{4}$. It now follows that $b = 1$ and $c = 1$.

b) In general, if (t, t) is the vertex, $-\frac{b}{2a} = t$, so $b = -2at$. We also know from the facts that (t, t) and $(t, -t)$ are on the parabola that

$$at^2 + bt + c = t \qquad (1)$$
$$at^2 - bt + c = -t \qquad (2)$$

By subtracting these equations, we get $2bt = 2t$, so $b = 1$ (always). Now $-2at = 1$, so $a = -\frac{1}{2t}$. Plugging into (1), we get $-\frac{1}{2t} \cdot t^2 + t + c = t$, which simplifies to $c = \frac{t}{2}$. So given specific t values, $a = -\frac{1}{2t},\ b = 1,\ c = \frac{t}{2}$. (This would also give us the specific solution from part a)). If we want to find a t value such that $a^2 + b^2 + c^2 = \frac{33}{16}$, we get $\frac{1}{4t^2} + 1 + \frac{t^2}{4} = \frac{33}{16}$; that is, $\frac{1}{t^2} + t^2 = \frac{17}{4}$ or $t^4 - \frac{17}{4}t^2 + 1 = 0$. This factors as $(t^2 - 4)(t^2 - \frac{1}{4}) = 0$, so our solutions are: $t = \pm 2, \pm\frac{1}{2}$.

c) $a^2 + b^2 + c^2 = \frac{1}{4t^2} + 1 + \frac{t^2}{4} = \frac{1}{4}(\frac{1}{t^2} + t^2) + 1$ is minimized when $f(t) = \frac{1}{t^2} + t^2$ is minimized. Note that $f(t) = f(\frac{1}{t})$ for all $t \neq 0$. We have $f(\pm 1) = 2$. Suppose $f(T) < 2$ for some value T. Then $T^2 f(T) = 1 + T^4 < 2T^2$, or $T^4 - 2T^2 + 1 = (T^2 - 1)^2 < 0$. This is impossible since $(T^2 - 1)^2 \geq 0$. Thus $t = \pm 1$ gives a minimum value of $\frac{1}{4}(2) + 1 = \frac{3}{2}$ for $a^2 + b^2 + c^2$.