BRITISH COLUMBIA SECONDARY SCHOOL MATHEMATICS CONTEST, 2018

Senior Final, Part A Problems & Solutions

1. There are n students in a gym class. Each is wearing a shirt (either red or blue) and shorts (also either red or blue). There are exactly 10 students wearing a red shirt, exactly 12 students wearing red shorts, and exactly 14 students wearing a shirt the same colour as their shorts. The smallest possible value of n is:

   (A) 16 (B) 18 (C) 20 (D) 22 (E) 24

   Solution

   If we have the largest overlap of red shirts and red shorts, then all 10 students wearing red shirts also have red shorts and red shorts. Then, there are $12 - 10 = 2$ students in red shorts and blue shirts, and $14 - 10 = 4$ students in both blue shirts and blue shorts. The answer is $10 + 2 + 4 = 16$.

   Answer: A

2. You have five cubes: one red, one yellow, one green, one light blue, and one dark blue. The number of ways in which the five cubes can be arranged in a single stack, without the blue cubes touching, is:

   (A) 24 (B) 48 (C) 72 (D) 96 (E) 120

   Solution

   There are $5! = 120$ ways to stack the cubes total. If the blue cubes are view as one cube, there are $4! = 24$ ways to stack the cubes. Given that there are 2 different orders for the blue cubes, there are $2 \times 24 = 48$ different stacks where the blue cubes occur together. Thus the number stacks where the blue cubes do not occur together is $120 - 48 = 72$.

   Answer: C

3. Suppose $f(x) = 4x$ for all real numbers x such that $97 \leq x < 103$. If $f(x + 6n) = f(x)$ for any real number x and any integer n, then the value of $f(2018)$ is:

   (A) 388 (B) 392 (C) 396 (D) 400 (E) 2018

   Solution

   Since $f(x) = f(x + 6n)$ for all real values x and integers n, the values of the function depend on the remainder when x is divided by 6. Noticing that $2018 = 6 \cdot 336 + 2$, we need to find a number between 97 and 103 with the same remainder of 2. Since $98 = 6 \cdot 16 + 2$, we obtain

   $$f(2018) = f(98) = 4 \cdot 98 = 392.$$

   Answer: B

4. Three cars travel at constant speeds over a long, straight track. If they start at the same time at the same end of the track, we find that car A finishes the track when car B has 120 metres to go and car C has 210 metres to go. When car B finishes, car C has 100 metres to go. The length of the track (in metres) is:

   (A) 1000 (B) 1050 (C) 1100

   (D) 1200 (E) not enough information given

Solution

Let x be the number of metres on the track. We have two points in time where we know the distances of the cars B and C as a function of x. Since cars B and C are both travelling at constant speeds, the distances they travel at given times are proportional. Thus

$$\frac{distance\ B\ has\ travelled\ when\ A\ finishes}{distance\ C\ has\ travelled\ when\ A\ finishes} = \frac{distance\ B\ has\ travelled\ when\ B\ finishes}{distance\ C\ has\ travelled\ when\ B\ finishes},$$

or $\frac{x-120}{x-210} = \frac{x}{x-100}$. Cross-multiplying and solving for x, we find $x = 1200m$.

Answer: D

5. Let $A(0,0)$, $B(2,2)$, $C(14,2)$, and $D(16,0)$ be four points on a circle with radius r. The value of r is:

(A) 8 (B) 10 (C) 12 (D) 15 (E) 20

Solution

The general equation of a (non-degenerate) circle is

$$x^2 + y^2 + ax + by + c = 0$$

where a, b, and c are real numbers uniquely determined by any three non-colinear points. Substituting the values of x and y from the first, second, and fourth points into the defining equation, we find $c = 0$, $a + b = -4$, and $a = 16$. From these equations we quickly deduce $b = -20$. Now we substitute these values in the equation and complete-the-squares on the two quadratics. We have

$$x^2 + y^2 + 16x - 20y = 0 \Leftrightarrow (x+4)^2 + (y-10)^2 = 10^2$$

We recognize this as the standard equation of a circle with radius 10 centred at $(-4, 10)$.

Answer: B

6. Suppose all 120 arrangements of the digits 1 to 5 are ordered in increasing size: starting with 12345, 12354, 12435, and continuing in order up to 54321. The 74th number is:

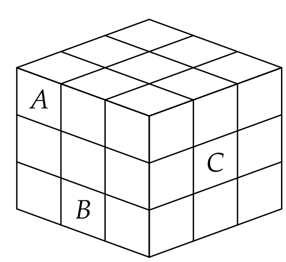
(A) 35412 (B) 35421 (C) 41235 (D) 41253 (E) 41325

Solution

If there are 120 arrangements of the digits 1 to 5 in order, then the first fifth $(120 \div 5) = 24$ must start with 1, the next 24 start with 2, and the next 24 start with 3 (a total of $24 \times 3 = 72$ so far). The 73rd and 74th must be the first two that start with 4, so the 73rd is 41235 and the 74th is 41253.

Answer: D

7. A large cube is made up of 27 smaller cubes, each of side length one centimetre, as shown in the diagram. If the three cubes marked A, B, and C are removed, then the total surface area (in cm^2) of the object that remains is:



(A) 66 (B) 60 (C) 58

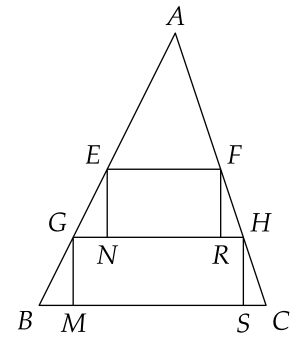
(D) 56 (E) 54

Solution

The surface area of the cube is 9 cm^3 per face; that is a total face area of $9 \times 6 = 54\ cm^2$. Removing A won't change the surface area, removing B with all 2 cm^2, and removing C will add 4 cm^2. So $54 + 6 = 60\ cm^2$.

Answer: B

8. Rectangles $FENR$ and $HGMS$ are inscribed in triangle ABC as shown. The area of triangle ABC is 60 and its base, BC, has length 10. If $EN = GM = 3$ then the sum of the areas of the two rectangles is:



(A) 30 (B) 32.5 (C) 36

(D) 36.5 (E) 37.5

Solution

The are of $\triangle ABC = 60$ and $\overline{BC}$ has length 10, so the height is 12. Since $\overline{EN} = \overline{GM} = 3$, the height of $\triangle AEF$ is 6, which is one half the height of $\triangle ABC$. The triangles are similar, so the ratios of their bases to their heights are the same. Thus the length of $\overline{EF}$ equals $\frac{1}{2}$ the length of $\overline{BC}$ which equals 5. Therefore, the area of $EFRN = 3 \cdot 5 = 15$. Since $\overline{GH}$ is halfway between $\overline{EF}$ (length = 5) and $\overline{BC}$ (length = 10), its total length is 7.5. Therefore, the area of $GHSM = 3 \cdot 7.5 = 22.5$ and the total area is $15 + 22.5 = 37.5$.

Answer: E

9. The smallest value of n for which the product

$$10^{1/2018} \times 10^{2/2018} \times 10^{3/2018} \times 10^{4/2018} \times \cdots \times 10^{n/2018}$$

exceeds $10,000$ is:

(A) 88 (B) 89 (C) 90 (D) 126 (E) 127

Solution

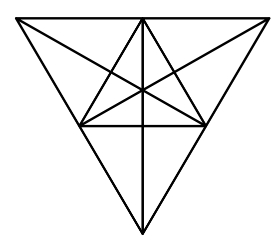
We want

$$10^{\frac{1}{2018}+\frac{2}{2018}+\cdots+\frac{n}{2018}} > 10^4.$$

So $\frac{1}{2018}(1+2+3+4+\cdots+n) > 4$, or $1+2+3+4+\cdots n > 8072$. The sum on the left contains $\frac{n}{2}$ pairs each adding to $n+1$. Thus $\frac{n}{2}(n+1) > 8072$. It follows that $n = 127$ is the smallest value for which this is true.

Answer: E

10. The total number of triangles that appear in the diagram is:



(A) 38 (B) 38 (C) 41

(D) 44 (E) 47

Solution

There are a number of systematic ways to classify and then count the triangles. It is a good idea to do it in two distinct ways in order to lessen the likelihood of a counting error.

One way to classify the triangles in the figure is by the number of distinct line segments crossing the interior of the triangle. This process obviously includes all of the triangles in the figure and does not include any of them twice. Careful examination of the figure reveals there are 12 triangles with no lines passing through them. Similarly, there are $12, 6, 9, 0, 0, 7$ triangle(s) for each of the cases of $1, \ldots, 6$ lines, and 1 for the final (non-zero) case Another way to classify the triangles is by the congruency class of the

triangle. To this end, for ease of identification, label all of the vertices of the triangles in the figure with the numbers $1, ..., 9$ beginning at top left and proceeding by rows from left to right so that 5 is in the middle and 9 is at the bottom. One can then label the triangles using ordered triples and classify them by their respective congruency classes. A complete list of representatives of the nine congruency classes is

$$(1,2,4),(1,2,5),(1,2,7),(1,2,9),(1,3,5)(1,3,10)(1,9,10),(2,5,4),(2,5,7).$$

Careful consideration of the figure reveals there are, respectively, $12, 6, 4, 6, 3, 1, 6, 6$, and 3 triangles in each class for a total of $12 + 6 + 4 + 6 + 3 + 1 + 6 + 6 + 3 = 47$ triangles.

Answer: E