BRITISH COLUMBIA SECONDARY SCHOOL MATHEMATICS CONTEST, 2018

Junior Final, Part B Problems & Solutions

1. The multiplication shown has all the digits missing except the digit 1 as shown. Assuming the digit 1 occurs only once, find all the missing digits. Be certain to clearly explain your reasoning.

$$\begin{array}{r} A\,B \\ \times\, C\,D \\ \hline E\,F \\ G\,1 \\ \hline H\,I\,J \end{array}$$

Solution

To get 1 where we carry out the multiplication $AB \times C = \underline{G1}$, where A is the tens digit and B is the unit digit. Since there are no other 1's digits, it follows that $A = 2$, $B = 7$, and $C = 3$; that is, $27 \times 3 = 81$. Thus $G = 8$. Now $27 \times D = EF$ (a 2-digit number) and consequently, $D \leq 3$. But clearly $D \neq 1, 3$ since neither D, E, F can be a 1. Thus $D = 2$. Now $EF = 2 \times 27 = 54$. Now $I = E + 1 = 6$, $J = F = 4$, and $H = G = 8$.

$$\begin{array}{r} 27 \\ \times 32 \\ \hline 54 \\ 81 \\ \hline 864 \end{array}$$

2. Bill is thinking of a 2-digit integer. When the number is divided by 3, the remainder is 2; when divided by 5 the remainder is 3, and when divided by 7 the remainder is 4. Find the number. Justify your answer.

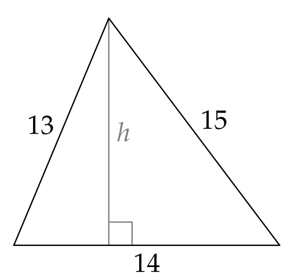
Solution

The possible 2-digit numbers (whose remainder when divided by 7 is 4) are:

$$11, 18, 25, 32, 39, 46, 53, 60, 67, 74, 81, 88, 95.$$

Since when the number is divided by 5, the remainder is 3, it must end in a 3 or 8, giving the possible numbers: $13, 18, 23, 28, 33, 38, \ldots$ etc. There are only the numbers $18, 53, 88$ in both lists. Finally, since the number has a remainder of 2 when divided by 3, only 53 satisfies this condition. Therefore the number is 53.

3. Consider a triangle with side lengths 13, 14, 15, and with height h, as shown in the diagram. Find the value of h and justify your answer.



Solution

Applying Pythagorus' theorem to the two right triangles, we obtain:

$$x^2 + h^2 = 13^2 \qquad (1)$$
$$(14 - x)^2 + h^2 = 15^2 \qquad (2)$$

Expanding (2), we have $196 - 28x + x^2 + h^2 = 225$. Subtracting (1) from this, we get $196 - 28x = 56$. Solving for x, we obtain $x = 5$. Therefore $h = \sqrt{13^2 - 5^2} = \sqrt{144} = 12$.

4. Find all possible sequences of consecutive positive integers that sum to 100, and explain why your list is complete.

Solution

There are two possible sequences: $18 + 19 + 20 + 21 + 22 = 100$ and $9 + 10 + 11 + 12 + 13 + 14 + 15 + 16 = 100$. To see this, let N be the number of integers in the sum and let $x, x+1, \ldots, x+N-1$ be the values in the sum. Let x_{av} denote the average value. Then $N \cdot x_{av} = 100$. Suppose N is odd. Then $x_{av} = x + \frac{N-1}{2}$. So $N \cdot (x + \frac{N-1}{2}) = 100$ and we see that N divides 100. Clearly $N \neq 1, 25$ and therefore $N = 5$. So $x + (\frac{5-1}{2}) = 20$, and hence $x = 18$. Therefore, when N is odd, we get the first sequence above. Suppose N is even. Then $x_{av} = \frac{x+x+(N-1)}{2} = \frac{2x+N-1}{2}$. Thus $N \cdot (2x + N - 1) = 200 = 2^3 \cdot 5^2$. Clearly, $N < 15$. Thus if 5 divides N, then $N = 2 \cdot 5 = 10$. We would then have $2x = 20 - 9 = 11$, which has no solution for x. Thus 5 does not divide N. Given that $2x + N - 1$ is an odd factor of 200, it follows that $2x + N - 1 = 25$ and hence $N = 8$. Consequently, $2x = 25 - 7 = 18$ and $x = 9$. This gives the second sequence.

5. Pick any five points on or inside a square of side length 2.

   (a) Explain why two of these five points must be separated by at *most* $\sqrt{2}$ units.

   (b) Find all sets of five points on this square so that each pair of points is separated by at *least* $\sqrt{2}$ units.

Solution

(a) Place the square on the two-dimensional plane so that the corners are located at $(-1, 1), (-1, -1), (1, -1)$, and $(1, 1)$. Then we see that the square splits into four equal "quadrants", with each quadrant being a small square of side length 1. By definition, each point on or inside the big square belongs to one of these four quadrants; some points may belong to more than one quadrant (e.g. the centre, which belongs to all four quadrants). Since we are selecting five points and there are four quadrants, at least two points must belong to the same quadrant. Consider two points which belong to the same quadrant. The farthest they can be apart is $\sqrt{2}$ units, which occurs when they are situated at opposite corners of the quadrant.

(b) Suppose five points (A, B, C, D, E) satisfy the condition so that each pair of points is separated by at least $\sqrt{2}$ units. By part (a), there must exist two points, say A and E, separated by exactly $\sqrt{2}$ units. By symmetry, let's assume that A and E are in the first quadrant. There are two cases: $A = (1, 1)$ and $E = (0, 0)$, or $A = (1, 0)$ and $E = (0, 1)$. In each case, we need to place three additional points so that each pair is separated by a distance at least $\sqrt{2}$ units. If $E = (0, 0)$ then we quickly see that the remaining four points must be $(\pm 1, \pm 1)$. If $A = (1, 0)$ and $E = (0, 1)$, then we see that the three remaining points would all have to belong to quadrant with corners $(0, 0), (-1, 0), (-1, -1), (0, -1)$, in which case at least one pair is distance less than $\sqrt{2}$ units apart. Thus, the only solution is if the set of five points is $(0, 0), (\pm 1, \pm 1)$.