BRITISH COLUMBIA SECONDARY SCHOOL MATHEMATICS CONTEST, 2017

Senior Preliminary Problems & Solutions

1. If $3^x + 3^x + 3^x = 243$ and $2^y + 2^y = 16$, what is the value of $x + y$?

(A) 6 (B) 7 (C) 8 (D) 9 (E) 10

Solution
Using the laws of exponents, we compute

$$3^x + 3^x + 3^x = 343 \Leftrightarrow 3 \cdot 3^x = 3^5 \Leftrightarrow 3^{x+1} = 3^5 \Leftrightarrow x + 1 = 5 \Leftrightarrow x = 4$$

and

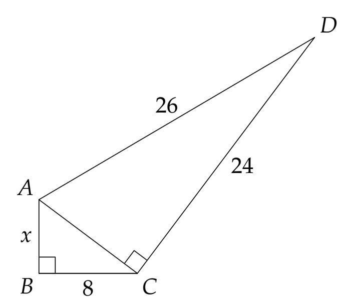
$$2^y + 2^y = 16 \Leftrightarrow 2 \cdot 2^y = 2^4 \Leftrightarrow 2^{y+1} = 2^4 \Leftrightarrow y + 1 = 4 \Leftrightarrow y = 3,$$

so $x + y = 4 + 3 = 7$.

The correct answer is (B).

Answer: B

2. In the figure shown, $ABCD$ is a quadrilateral with $\angle ABC = 90°$ and $\angle ACD = 90°$. If $BC = 8$, $CD = 24$, $AD = 26$ and $AB = x$, determine x.



(A) 6 (B) 8 (C) 10

(D) 12 (E) 15

Solution
We apply Pythagoras' theorem to the two right triangles shown to obtain the equations

$$x^2 + 8^2 = AC^2 \text{ and } AC^2 + 24^2 = 26^2$$

We have

$$x^2 + 8^2 = 26^2 - 24^2 = (26 + 24)(26 - 24) = 100,$$

so $x^2 = 100 - 64 = 36$ and $x = 6$.

The correct answer is (A).

Answer: A

3. A rectangular box has six faces, whose areas are 45, 45, 90, 90, 200 and 200 cm^2. Determine the volume of this box.

(A) 450 cm^3 (B) 600 cm^3 (C) 810 cm^3 (D) 900 cm^3 (E) 1800 cm^3

Solution
Let $x \le y \le z$ denote the three lengths of the edges, and let V denote the volume. We are given $xy = 45$, $xz = 90$, and $yz = 200$. Multiplying these together gives

$$(xy)(xz)(yz) = (xyz)^2 = 45 \cdot 90 \cdot 200 = 2^4 \cdot 3^4 \cdot 5^4,$$

so $V = xyz = \sqrt{2^4 \cdot 3^4 \cdot 5^4} = 2^2 \cdot 3^2 \cdot 5^2 = 900$.

The correct answer is (D).

Answer: D

4. The average age of 120 people is 35. The average age of the men is 32, while the average age of the women is 37. How many women are there?

(A) 60 (B) 66 (C) 72 (D) 75 (E) 80

Solution

For $k = 1 \ldots 120$, let x_k denote the age of the k^{th} person (with the men listed first), and let m and w denote, respectively, the number of men and women. We are given

$$\frac{x_1 + x_2 + \cdots + x_{120}}{120} = 35$$

$$\frac{x_1 + x_2 + \cdots + x_m}{m} = 32$$

$$\frac{x_{m+1} + \cdots + x_{120}}{w} = 37$$

$$m + w = 120$$

or, equivalently,

$$x_1 + x_2 + \cdots + x_{120} = 35 \cdot 120$$

$$x_1 + x_2 + \cdots + x_m = 32m$$

$$x_{m+1} + \cdots + x_{120} = 37w$$

$$m + w = 120$$

Adding the second and third equations, we see $32m + 37w = 35 \cdot 120$. Combining this with the fourth equation quickly yields $w = 72$.

Alternate solution:

You can think of the average as a "balance point," so the total amount that higher individual data values go above the average is equal to the total amount that the lower individual data go below the average. The men on average are 3 points below the overall average and the women on average are 2 points above the overall average, so $3m = 2w$. Since the total number of people $m + w = 120$, we find $m = 72$ and $w = 48$.

The correct answer is (C).

Answer: C

5. Four students each roll a 6-sided die (assume the numbers 1 through 6 are equally likely to occur). What is the probability that they all roll different numbers?

(A) less than 0.15 (B) between 0.15 and 0.3 (C) between 0.3 and 0.45

(D) between 0.45 and 0.6 (E) greater than 0.6

Solution

Since there are six possible outcomes from each roll, we see the number of ways to roll any sequence of four numbers from 1 to 6 is $6 \cdot 6 \cdot 6 \cdot 6$. To roll four different numbers in sequence, the first person has six choices; the second five choices, and so on for a total of $6 \cdot 5 \cdot 4 \cdot 3$ sequences. By definition, the probability P of rolling such a sequence is then

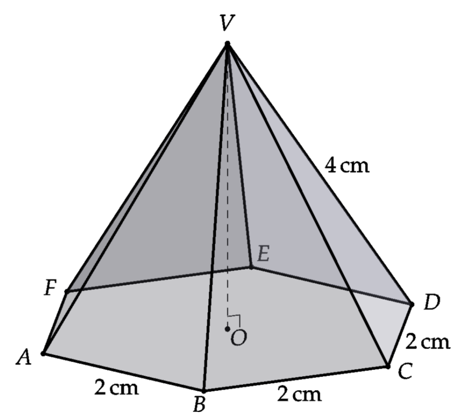
$$P = \frac{6 \cdot 5 \cdot 4 \cdot 3}{6 \cdot 6 \cdot 6 \cdot 6} = \frac{5}{18}.$$

One step of the division algorithm shows $\frac{5}{18}$ is slightly larger than 0.2.

The correct answer is (B).

Answer: B

6. The base of a certain pyramid is a regular hexagon with side length 2 cm. Each of the sloped edges has length 4 cm as shown in the diagram. Determine the angle AVD.



(A) 30° (B) 45° (C) 60°

(D) 75° (E) 90°

Solution

Since the base is a regular hexagon, $\triangle OAB$ is equilateral so we have $AO = 2$ cm. Thus $AD = 4$ cm. Now we can see triangle AVD is equilateral, so angle AVD is 60 degrees.

The correct answer is (C).

Answer: C

7. Recall that $n! = n \cdot (n-1)(n-2)\cdots 2 \cdot 1$. For example, $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$. Which of the following numbers is a perfect square?

(A) $\frac{45!46!}{2}$ (B) $\frac{46!47!}{2}$ (C) $\frac{47!48!}{2}$ (D) $\frac{48!49!}{2}$ (E) $\frac{49!50!}{2}$

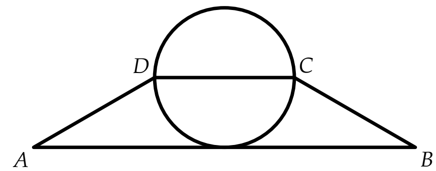
Solution

Each of the five choices can be placed in the form $\frac{[(n-1)!)]^2 \cdot n}{2}$ for some integer, n, so it remains to find the one for which $\frac{n}{2}$ is a perfect square. We calculate $\frac{48}{2} = 24$ is not a perfect square; $\frac{49}{2}$ is not even an integer let alone a perfect square; and $\frac{50}{2} = 25 = 5^2$ is a perfect square.

The correct answer is (E).

Answer: E

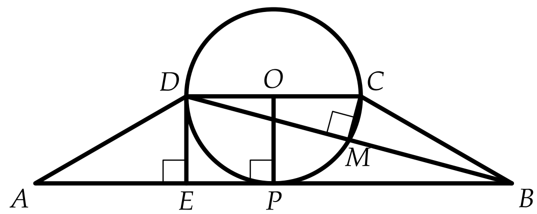
8. Consider a trapezoid $ABCD$ with AB parallel to CD. A circle has a diameter CD, passes through the midpoints of the diagonals AC and BD, and touches AB at one point. The angle DAB is:



(A) $15°$ (B) $22.5°$ (C) $30°$

(D) $36°$ (E) $45°$

Solution



M is the midpoint of the diagonal DB so M is on the circle, and so $\angle DMC = 90°$. But then the right triangles DMC and BMC are congruent, so that $DC = CB$. Similarly, $AD = DC$.

Let P be the point of tangency of AB and the circle, and let E be the foot of the perpendicular from point D to AB. Let O be the centre of the circle. Then

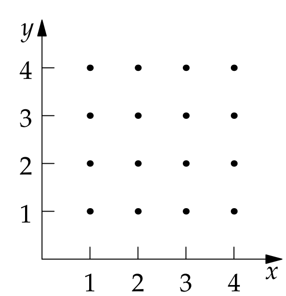
$$DE = OP = OD = OC = \frac{1}{2}DA.$$

Thus $DH = \frac{1}{2}DA$. It follows that $\angle DAB = 30°$. (To show this, reflect triangle ADE in the line AB to produce triangle ADF. Evidently this triangle is equilateral.)

The correct answer is (C).

Answer: C

9. In the xy-plane, consider the sixteen points (x, y) with x and y both integers such that $1 \le x \le 4$ and $1 \le y \le 4$ (as shown in the diagram). Determine the number of triangles with positive area whose three vertices are chosen from these sixteen points.



(A) 496 (B) 516 (C) 520

(D) 528 (E) 560

Solution

Since a triangle with positive area is completely determined by any three non-collinear points, it is sufficient to count the number of ways of choosing three non-collinear points from the grid of sixteen points. The easiest way to count them is to use the so-called inclusion-exclusion principle: first we include all possible combinations of three points in the count; and then we exclude from the count those combinations which are collinear.

Reasoning in an elementary manner (or, if you know it, using the standard formula $C(n,k) = \frac{n!}{k!(n-k)!}$), we determine the number of ways of choosing three points from sixteen is $T = \frac{16 \cdot 15 \cdot 14}{3 \cdot 2 \cdot 1} = 560$. Next, we count the triples of collinear points in the grid. In the first place, we can find three collinear points in the grid by removing any one of the four points in any of the ten rows, columns, or main diagonals.

This gives $4 \cdot 10 = 40$ such triples of collinear points. In second place, we can find four additional triples of collinear points by taking all three points in each of the four sub-diagonals in the grid. This gives a total of $40 + 4 = 44$ triples of collinear points and, hence, a total of $560 - 44 = 516$ triangles with positive area in the grid.

The correct answer is (B).

Answer: B

10. What is the number of pairs of positive integers (p, q) such that the equation $x^2 - px - q = 0$ has a root x with $0 < x < 8$?

(A) 0 (B) 64 (C) 162 (D) 210 (E) 217

Solution

The positive root of this quadratic is
$$x = \frac{p + \sqrt{p^2 + 4q}}{2}.$$
We want to find the number of pairs (p, q) such that
$$\frac{p + \sqrt{p^2 + 4q}}{2} < 8.$$
This inequality simplifies to
$$q < 64 - 8p.$$
Since q is positive, then $p < 8$; also $p > 0$, and we can now list all the pairs that meet the above inequality:

If $p = 1$, then $q = 1, 2, \ldots, 55$.

If $p = 2$, then $q = 1, 2, \ldots, 47$.

If $p = 3$, then $q = 1, 2, \ldots, 39$.

...

If $p = 7$, then $q = 1, 2, \ldots, 7$.

Then the number of all such pairs (p, q) is
$$7 + 15 + \cdot + 47 + 55 = \frac{7 + 55}{2} \cdot 7 = 217.$$

The correct answer is (E).

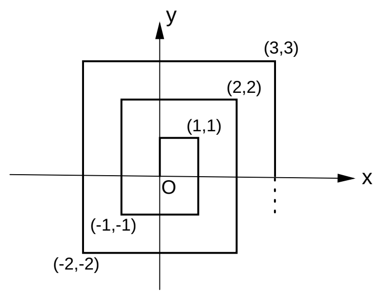
Alternate solution: The quadratic formula tells us the roots of the equation are
$$\tfrac{p \pm \sqrt{(-p)^2 - 4(1)(-q)}}{2} = \tfrac{p \pm \sqrt{p^2 + 4q}}{2}$$
We must count the number of pairs of positive integers (p, q) such that
$$0 \leq \tfrac{p \pm \sqrt{p^2 + 4q}}{2} \leq 8.$$
Since everything is positive, we can square away the root and rearrange to obtain the equivalent inequality $p^2 + q \leq 64$. Listing all possible solutions of this inequality, as in the first solution, we count 217 possible pairs (p, q).

Answer: E

11. A particle moves through the the xy-plane in the following spiral formation: it starts at (0,0), then it moves 1 unit parallel to an axis in each second. The first several moves are (0,1), (1,1), (1,0), (1,-1), (0,-1), (-1,-1), (-1,0), (-1,1), (-1,2), (0,2), (1,2), (2,2), (2,1), and so on. At which point will the particle be after exactly 2017 seconds ?

(A) $(-22, -22)$ (B) $(-22, -15)$ (C) $(-22, -5)$ (D) $(-22, 5)$ (E) $(-22, 15)$

Solution



We observe that the lengths traveled before each change of direction form the sequence

$$1, 1, 2, 2, 3, 3, 4, 4, 5, 5, 6, 6, \ldots, n, n, \ldots,$$

and that after every two segments of equal length traveled the particle is in one of the corner points of its path located on the bisector of the first (and the third) quadrant. The sum of the first $2n$ terms of this sequence is $2(1 + 2 + \cdots + n) = n(n + 1)$, so that, more precisely:

If n is odd, then, after $n(n + 1)$ seconds, the particle will be at the point $\left(\lceil \frac{n}{2} \rceil\right), \left(\lceil \frac{n}{2} \rceil\right)$;

If n is even, then, after $n(n + 1)$ seconds, the particle will be at the point $\left(-\frac{n}{2}, -\frac{n}{2}\right)$.

Now, $44(45) = 1980 < 2017 < 2070 = 45(46)$, and 44 is even, so that the particle will be at the point $(-22, -22)$ in 1980 seconds, and then it will travel up (in the next 45 seconds). The point that corresponds to 2017 seconds of traveling will be visited 37 seconds later, so that its second coordinate will be $-22 + 37 = 15$. Hence the point of interest is $(-22, 15)$ and the correct answer is (E).

$\boxed{\textbf{Answer: E}}$

12. If $a^2 + b^2 = c^2 + d^2 = 1$ and $ac + bd = 0$ then the maximum possible value of $ad - bc$ is:

(A) 0 (B) $\frac{1}{2}$ (C) 1 (D) $\sqrt{2}$ (E) 2

Solution
The (somewhat non-obvious) algebraic identity

$$(a^2 + b^2)(c^2 + d^2) = (ac + bd)^2 + (ad - bc)^2$$

is easily established by multiplying out both sides. We are given $a^2 + b^2 = c^2 + d^2 = 1$, and $ac + bd = 0$. It follows the maximum possible value of $ad - bc$ is 1.

The correct answer is (C).

Alternate solution:
Let

$$a = \sin x, \quad b = \cos x, \quad c = \sin y, \quad d = \cos y$$

for some angles x and y. Then

$$ac + bd = \sin x \sin y + \cos x \cos y = \cos(x - y) = 0 \qquad \text{(difference identity for cos).}$$

But since

$$\sin^2(x - y) + \cos^2(x - y) = 1 \qquad \text{(Pythagorean identity)}$$

this implies $\sin(x - y) = 1$ or -1. Since

$$ad - bc = \sin x \cos y - \cos x \sin y = \sin(x - y) \qquad \text{(difference identity for sin)}$$

we see that the maximum value of $ad - bc = \sin(x - y)$ must be 1.

Answer: C