BRITISH COLUMBIA SECONDARY SCHOOL MATHEMATICS CONTEST, 2017

Senior Final, Part A Problems & Solutions

1. Let x, y, z be real numbers such that $x + y - z = 2$, $x - y + z = 4$, and $-x + y + z = 6$. Determine the value of $x + y + z$.

   (A) 6 (B) 8 (C) 10 (D) 12 (E) 24

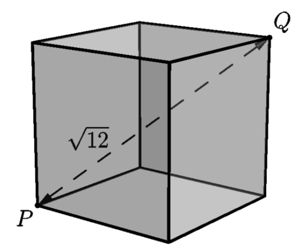
   Solution
   We add the three given equations $x + y - z = 2, x - y + z = 4, -x + y + z = 6$ and quickly find $x + y + z = 12$.

   The correct answer is (D).

   Alternate solution: The system can be solved, with $x = 3$, $y = 4$, $z = 5$. This gives $x + y + z = 12$.

   Answer: D

2. A cube has diagonal PQ with length $\sqrt{12}$ as shown. Determine the volume of the cube.



   (A) 8 (B) 12 (C) $12\sqrt{2}$

   (D) 27 (E) $12\sqrt{2}$

   Solution
   Let x be the edge length of the cube. Applying Pythagoras' theorem to the right-triangle on the base of the cube, we find the diagonal of the base has length $L = \sqrt{x^2 + x^2} = x\sqrt{2}$. Now, by another application of Pythagoras theorem, this time to the right-triangle having PQ as its hypotenuse, we find $x^2 + (x\sqrt{2})^2 = 3x^2 = 12$, so $x = 2$, and the volume of the cube is $V = x^3 = 2^3 = 8$.

   The correct answer is (A).

   Answer: A

3. Alice is driving to Bob's house, intending to arrive at a certain time. If she drives at 60 km/h she will arrive 5 minutes late. If she drives at 90 km/h she will arrive 5 minutes early. If she drives at x km/h she will arrive exactly on time. Determine x.

   (A) 66 (B) 70 (C) 72 (D) 75 (E) 78

   Solution
   Let D, and V denote distance and velocity, respectively. Let T denote the exact time and let x denote the velocity corresponding to that time. Noting that 5 minutes is $\frac{1}{12}$ of an hour, for the late journey we have $\frac{D}{60} = T + \frac{1}{12}$; for the early journey we have $\frac{D}{90} = T - \frac{1}{12}$; and for the exact journey we have $\frac{D}{x} = T$. From these equations, we easily deduce $x = 72$.

   The correct answer is (C).

   Answer: C

4. Anya and Bert play a game where they flip a coin that is equally likely to come up heads or tails. They take turns flipping the coin, with Anya going first. This first person to flip tails wins. Determine the probability that Anya wins the game.

(A) $\frac{1}{2}$ (B) $\frac{2}{3}$ (C) $\frac{3}{5}$ (D) $\frac{3}{4}$ (E) $\frac{5}{6}$

Solution

Let T denote tails, and let H denote heads. Suppose Anya rolls, but the result of her roll is revealed only after Bert has rolled. The four equally likely outcomes in the sample space are TT, TH, HT, and HH. If the fourth outcome occurs, then the game is a draw, so they play again. We see the outcome HH is not relevant to the probability of Anya winning or losing, so we are in fact looking for the conditional probability that Anya wins the game given that outcome HH did not occur. Clearly, two of the remaining three outcomes are favourable to Anya, so the probability Anya wins is $\frac{2}{3}$.

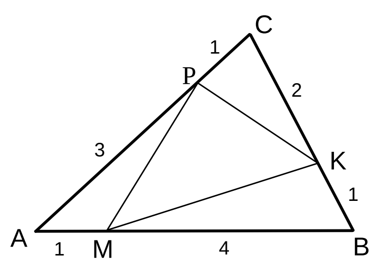
The correct answer is (B).

$\boxed{\textbf{Answer: B}}$

5. The area of $\triangle ABC$ is 1. Points M, K and P are on the segments AB, BC and CA, respectively, so that $AM = \frac{1}{5}AB$, $BK = \frac{1}{3}BC$, and $CP = \frac{1}{4}CA$. The area of $\triangle MKP$ is

(A) $\frac{5}{12}$ (B) $\frac{1}{2}$ (C) $\frac{7}{12}$ (D) $\frac{11}{15}$ (E) $\frac{13}{20}$

Solution



We have

$$[\text{area of } AMP] = \frac{[\text{area of } AMP]}{[\text{area of } ABC]} = \frac{AM}{AB} \cdot \frac{AP}{AC} = \frac{1}{5} \cdot \frac{3}{4} = \frac{3}{20}$$

$$[\text{area of } CPK] = \frac{[\text{area of } CPK]}{[\text{area of } ABC]} = \frac{CP}{CA} \cdot \frac{CK}{CB} = \frac{1}{4} \cdot \frac{2}{3} = \frac{1}{6}$$

$$[\text{area of } BKM] = \frac{[\text{area of } BKM]}{[\text{area of } ABC]} = \frac{BK}{BC} \cdot \frac{BM}{BA} = \frac{1}{3} \cdot \frac{4}{5} = \frac{4}{15}.$$

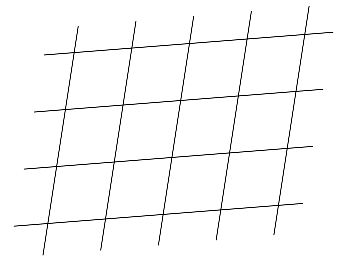
Thus

$$\begin{aligned}[\text{area of } PMK] &= [\text{area of } ABC] - ([\text{area of } AMP] + [\text{area of } CPK] + [\text{area of } BKM]) \\ &= 1 - \left(\frac{3}{20} + \frac{1}{6} + \frac{4}{15}\right) = \frac{5}{12}\end{aligned}$$

so the correct answer is (A)

$\boxed{\textbf{Answer: A}}$

6. Five parallel lines are drawn, and then four other parallel lines are drawn in a different direction. How many distinct parallelograms are there in the picture?



(A) 30 (B) 45 (C) 52

(D) 60 (E) 100

Solution

There are $\binom{5}{2} = 10$ ways to choose two parallel sides of a parallelogram from the first direction and $\binom{4}{2} = 6$ ways to choose two parallel sides from the other direction. Hence the number of parallelograms is $10(6) = 60$; the correct answer is (D). (If the students do not know the combinations formula, they can still count like: For the first direction, there are 5 ways to choose the first of two parallel sides and 4 ways to choose the second, which gives a total count of 20, but this has to be divided by two, because the order of the two parallel sides does not matter.)

Alternative solution:

We proceed through a systematic count as follows. Call a parallelogram type (m, n) if it contains m rows and n columns of the basic (that is, smallest) type of parallelogram. The types and number of each type are

$$\begin{array}{llll} (1,1):12 & (1,2):9 & (1,3):6 & (1,4):3 \\ (2,1):8 & (2,2):6 & (2,3):4 & (2,4):2 \\ (3,1):4 & (3,2):3 & (3,3):2 & (3,4):1 \end{array}$$

It follows the total number of parallelograms in the figure is $12+8+4+9+6+3+6+4+2+3+2+1 = 60$.

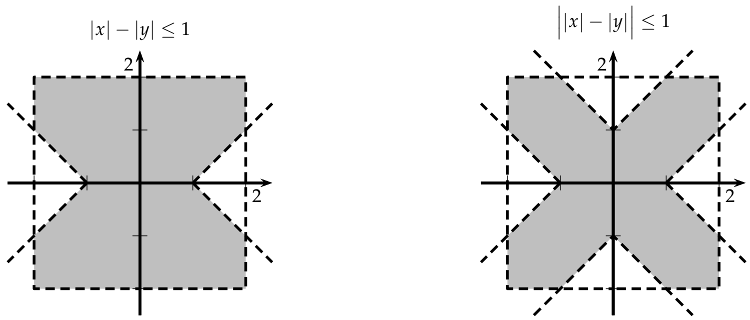
$\boxed{\textbf{Answer: D}}$

7. Let $|a|$ be the absolute value of the number a. The points (x, y) on the coordinate plane satisfying $|x| \le 2$, $|y| \le 2$, and $\big||x| - |y|\big| \le 1$ define a region with area:

(A) 8 (B) 10 (C) 12 (D) 14 (E) 16

Solution

The two inequalities $|x| \le 2$ and $|y| \le 2$ define the square in the xy-plane with $-2 \le x \le 2$ and $-2 \le y \le 2$. The inequality $|x| - |y| \le 1$ defines the region that is either above the x-axis and the lines $y = x - 1$ and $y = -x - 1$ or below the x-axis and the lines $y = x + 1$ and $y = -x + 1$, as shown on the left below. The inequality $\big||x| - |y|\big| \le 1$ defines the region obtained from that on the left by interchanging the roles of x and y, as shown on the right below.

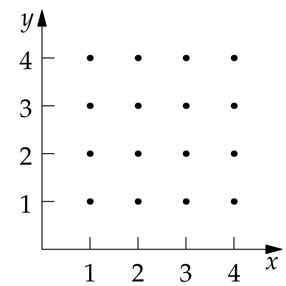


The required area is the shaded region inside the square in the diagram on the right above. The area of the square is $4 \times 4 = 16$. The dart on each of the four sides has area 1. So the required area is $16 - 4 = 12$.

Answer: C

8. In the xy-plane, consider the sixteen points (x, y) with x and y both integers such that $1 \le x \le 4$ and $1 \le y \le 4$ (as shown in the diagram). Determine the number of ways we can label ten of these points $A, B, C, D, E, F, G, H, I, J$ such that the nine distances AB, BC, CD, DE, EF, FG, GH, HI, IJ satisfy the inequality

$$AB < BC < CD < DE < EF < FG < GH < HI < IJ.$$



(A) 4 (B) 8 (C) 12 (D) 24 (E) 36

Solution

Let (x_1, y_1) and (x_2, y_2) be any two of the points. Then the distance between those two points is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. As $1 \le x_1, x_2, y_1, y_2 \le 4$, these distances can take only the following values:

$$\sqrt{1}, \sqrt{2}, \sqrt{4}, \sqrt{5}, \sqrt{8}, \sqrt{9}, \sqrt{10}, \sqrt{13}, \sqrt{18}.$$

Notice there are nine distinct values here. Thus, if $AB < BC < CD < DE < EF < FG < GH < HI < IJ$ then we must have $IJ = \sqrt{18}$, i.e. the x and y coordinates of I and J must both differ by 3. Therefore, J can only be one of $(1, 1)$, $(1, 4)$, $(4, 1)$, $(4, 4)$. Once J is determined, I is forced.

For example, if J is $(1, 1)$, then I must be $(4, 4)$. Let's consider this choice for J, as the other three cases are equivalent by rotating the figure 90 degrees clockwise.

$HI = \sqrt{13}$ implies H is $(2, 1)$ or $(1, 2)$. Let's consider the case $H = (2, 1)$, as the other case is equivalent by reflecting the points along the line $y = x$. So we have $J = (1, 1)$, $I = (4, 4)$, and $H = (2, 1)$. From above, we see there are $4 \times 2 = 8$ distinct configurations for points J, I, H, all of which are symmetric/equivalent.

$GH = \sqrt{10}$ implies $G = (1, 4)$ or $(3, 4)$. But $G = (1, 4)$ leads to a contradiction as $FG = \sqrt{9}$ forces F to be either I or J, which we have already chosen. Thus, $G = (3, 4)$.

$FG = \sqrt{9}$ implies $F = (3, 1)$.

$EF = \sqrt{8}$ implies $E = (1, 3)$.

$DE = \sqrt{5}$ implies $D = (3, 2)$.

$CD = \sqrt{4}$ implies $C = (1, 2)$.

$BC = \sqrt{2}$ implies $B = (2, 3)$.

$AB = \sqrt{1}$ implies A can be any of $(2, 2)$, $(3, 3)$, $(2, 4)$, i.e., any of three choices.

Therefore, there are $4 \times 2 \times 3 = 24$ distinct ways we can label ten points to satisfy the desired inequality.

Answer: D

9. There are two integers n such that $\dfrac{n^2 - 71}{7n + 55}$ is a natural number. The sum of these two integers is

(A) -21 (B) 13 (C) 32 (D) 49 (E) 98

Solution

The quotient will certainly be an integer when the denominator is ± 1. We can quickly see $n = -8$ produces such a denominator. From the given choices we know the other value of n must be one of

$$\begin{aligned} -21 + 8 &= -13 \\ 13 + 8 &= 21 \\ 32 + 8 &= 40 \\ 49 + 8 &= 57 \\ 98 + 8 &= 116. \end{aligned}$$

Of these, only $n = 57$ gives an integer value fo the given quotient. So the sum of $-8 + 57 = 49$.

The correct answer is (D).

Alternate solution: Let $\frac{n^2-71}{7n+55} = k$ be an integer. Then

$$n^2 - 7kn - (55k + 71) = 0,$$

so that

$$n = \frac{7k \pm \sqrt{49k^2 + 4(55k + 71)}}{2}.$$

Since n is an integer, $D = 49k^2 + 220k + 284$ must be a square of an integer. (Otherwise, if D is not a perfect square, then $\sqrt{D}$ is irrational, and so is n.) Observe that

$$49k^2 + 210k + 225 < D < 49k^2 + 238k + 289$$

or

$$(7k + 15)^2 < D < (7k + 17)^2.$$

If D is a square of an integer, then the only possibility is $D = (7k + 16)^2$. Thus $49k^2 + 220k + 284 = (7k + 16)^2$, which gives $k = 7$. Then $n^2 - 49n - 456 = 0$, which gives two possible values of n, 57 and -8. The sum of these n is 49.

Answer: D

10. Let ABC be an acute-angled triangle with $\cos A = 1/50$. The point O is the centre of the circumcircle of triangle ABC, and I is the centre of the incircle of triangle ABC. Determine the maximum possible value of AI/AO.

(A) $\frac{3}{5}$ (B) $\frac{3}{4}$ (C) 1 (D) $\frac{4}{3}$ (E) $\frac{5}{3}$

Solution

AOC is an isosceles triangle with $OA = OC$. Because O is the centre of the circumcircle, angle $AOC = 2B$, from which we get angle $OAC = 90° - B$. Because I is the incentre of the triangle, AI is an angle bisector, from which we get angle $BAI = A/2$ and angle $ABI = B/2$.

By the Sine Law, we have the following equations:

$$\begin{aligned} AI/AB &= \sin(B/2)/\sin(180° - (A + B)/2) \\ AB/AO &= 2\sin C = 4\sin(C/2)\cos(C/2). \end{aligned}$$

Multiplying these three equations, and using the identities

$$\cos(90° - x) = \sin(x), \qquad \sin(2x) = 2\sin x \cos x$$

we obtain

$$AI/AO = 4\sin(B/2)\sin(C/2).$$

Using the identities

$$\cos(x+y) = \cos x \cos y - \sin x \sin y, \qquad \cos(x-y) = \cos x \cos y + \sin x \sin y$$

we see that

$$\begin{aligned} AI/AO &= 4\sin(B/2)\sin(C/2) = 2\cos(B/2 - C/2) - 2\cos(B/2 + C/2) \\ &= 2\cos(B/2 - C/2) - 2\cos((180^\circ - A)/2) \\ &= 2\cos(B/2 - C/2) - 2\sin(A/2). \end{aligned}$$

To maximize AI/AO, clearly we want $B = C$, i.e., for triangle ABC to be isosceles.
Our desired maximum value is $2 \times 1 - 2\sin(A/2) = 2 - 2\sin(A/2)$.
Since $\cos A = 1/50$, and A is an acute angle, we have

$$1 - 2\sin^2(A/2) = 1/50 \qquad \text{or } 49/50 = 2\sin^2(A/2) \qquad \text{or } \sin(A/2) = 7/10.$$

Therefore, the maximum value of AI/AO is $2 - 2\sin(A/2) = 2 - 2 \times 7/10 = 3/5$.

Answer: A