BRITISH COLUMBIA SECONDARY SCHOOL MATHEMATICS CONTEST, 2017

Junior Preliminary Problems & Solutions

1. If x is a number larger than 5, which of the following expressions is the smallest?

   (A) $5/(x-1)$ (B) $5/x$ (C) $5/(x+1)$ (D) $x/5$ (E) $(x+1)/5$

   Solution
   We have $0 < x-1 < x < x+1$. Taking reciprocals and multiplying by 5 gives
   $$\frac{5}{x+1} < \frac{5}{x} < \frac{5}{x-1}$$
   to eliminates (A) and (B).
   Similarly, from $x < x+1$ we can deduce
   $$\frac{x}{5} < \frac{x+1}{5}$$
   to eliminate (E). So the answer must be (C) or (D).

   Since $x > 5$ we have
   $$\frac{5}{x+1} < 1 < \frac{x}{5}$$
   so the correct answer is (C).

   Answer: C

2. If $3^x + 3^x + 3^x = 243$, determine the value of x.

   (A) $\frac{4}{3}$ (B) $\frac{5}{3}$ (C) 3 (D) 4 (E) 5

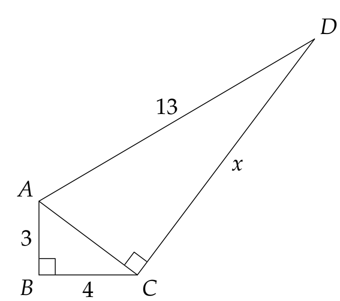
   Solution
   Using the laws of exponents, we compute
   $$3^x + 3^x + 3^x = 3 \cdot 3^x = 3^{x+1} = 243 = 3^5 \Leftrightarrow x+1 = 5 \Leftrightarrow x = 4.$$
   The correct answer is (D).

   Answer: D

3. In the figure shown, $ABCD$ is a quadrilateral with $\angle ABC = 90°$ and $\angle ACD = 90°$. If $AB = 3$, $BC = 4$, $AD = 13$ and $CD = x$, determine x.

   (A) 5 (B) 10 (C) 12

   (D) $\sqrt{194}$ (E) none of these

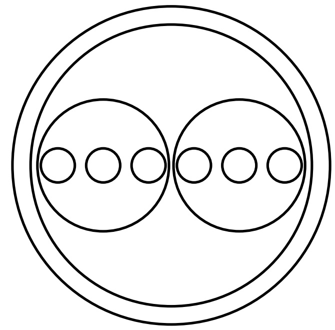


   Solution
   We apply Pythagoras' theorem twice. On the smaller right-triangle, we have $3^2 + 4^2 = AC^2$, so $AC = 5$. On the larger right-triangle, we have $AC^2 + x^2 = 5^2 + x^2 = 13^2$, so $x = 12$.

   The correct answer is (C).

   Answer: C

4. There are 10 circles in the diagram shown. How many ways are there to label three of the circles, with labels A, B, and C, so that circle A is inside circle B, which is inside circle C?



(A) 6 (B) 8 (C) 12
(D) 14 (E) 20

Solution
If we label the four different-sized circles from largest to smallest using the labels $4, 3, 2$, and 1, then the label sequences satisfying the conditions of the problem are $(4, 3, 2)$, $(4, 3, 1)$, $(4, 2, 1)$ and $(3, 2, 1)$. There are two label sequences of the first type, and six of each of the second, third and fourth types. Thus there are $2 + 6 + 6 + 6 = 20$ ways to choose three circles so that the second is inside the first, and the third is inside the second.

The correct answer is (E).

Answer: E

5. All of the edge lengths of a cube are doubled. By what percentage does the volume of the cube increase?

(A) 100 (B) 200 (C) 300 (D) 400 (E) 700

Solution
If x is the side length of the original cube, then the enlarged cube has side length $2x$ and their respective volumes are

$$V_1 = x^3$$

and

$$V_2 = (2x)^3 = 8x^3 = 8V_1$$

The difference in volume is

$$V_2 - V_1 = 8V_1 - V_1 = 7V_1,$$

which amounts to a 700 percent increase.

The correct answer is (E).

Answer: E

6. A rectangular box has six faces, whose areas are 5, 5, 10, 10, 18 and 18 cm^2. Determine the volume of this box.

(A) $15\,\text{cm}^3$ (B) $30\,\text{cm}^3$ (C) $45\,\text{cm}^3$ (D) $60\,\text{cm}^3$ (E) $90\,\text{cm}^3$

Solution
Let $x \le y \le z$ denote the three lengths of the edges, and let V denote the volume. We are given $xy = 5$, $xz = 10$, and $yz = 18$. Multiplying these together, we have

$$(xy)(xz)(yz) = (xyz)^2 = 5 \cdot 10 \cdot 18 = 2^2 \cdot 3^2 \cdot 5^2,$$

so $V = xyz = \sqrt{2^2 \cdot 3^2 \cdot 5^2} = 2 \cdot 3 \cdot 5 = 30$.

The correct answer is (B).

Answer: B

7. The average age of 120 people is 35. The average age of the men is 32, while the average age of the women is 37. How many women are there?

(A) 60 (B) 66 (C) 72 (D) 75 (E) 80

Solution

For $k = 1 \dots 120$, let x_k denote the age of the k^{th} person (with the men listed first), and let m and w denote, respectively, the number of men and women. We are given

$$\frac{x_1 + x_2 + \cdots + x_{120}}{120} = 35$$

$$\frac{x_1 + x_2 + \cdots + x_m}{m} = 32$$

$$\frac{x_{m+1} + \cdots + x_{120}}{w} = 37$$

$$m + w = 120$$

or, equivalently,

$$x_1 + x_2 + \cdots + x_{120} = 35 \cdot 120$$

$$x_1 + x_2 + \cdots + x_m = 32m$$

$$x_{m+1} + \cdots + x_{120} = 37w$$

$$m + w = 120$$

Adding the second and third equations, we see $32m + 37w = 35 \cdot 120$. Combining this with the fourth equation quickly yields $w = 72$.

Alternate solution:

You can think of the average as a "balance point," so the total amount that higher individual data values go above the average is equal to the total amount that the lower individual data go below the average. The men on average are 3 points below the overall average and the women on average are 2 points above the overall average, so $3m = 2w$. Since the total number of people $m + w = 120$, we find $m = 72$ and $w = 48$.

The correct answer is (C).

Answer: C

8. Four students each roll a 6-sided die (assume the numbers 1 through 6 are equally likely to occur). What is the probability that they all roll different numbers?

(A) less than 0.15 (B) between 0.15 and 0.3 (C) between 0.3 and 0.45

(D) between 0.45 and 0.6 (E) greater than 0.6

Solution

Since there are six possible outcomes from each roll, we see the number of ways to roll any sequence of four numbers from 1 to 6 is $6 \cdot 6 \cdot 6 \cdot 6$. To roll four different numbers in sequence, the first person has six choices; the second five choices, and so on for a total of $6 \cdot 5 \cdot 4 \cdot 3$ sequences. By definition, the probability P of rolling such a sequence is then

$$P = \frac{6 \cdot 5 \cdot 4 \cdot 3}{6 \cdot 6 \cdot 6 \cdot 6} = \frac{5}{18}.$$

One step of the division algorithm shows $\frac{5}{18}$ is slightly larger than 0.2.

The correct answer is (B).

Answer: B

9. A group of strangers attended a party. Each person shook hands with everyone else. Unfortunately, Chris arrived late and was able to shake hands with only some of the guests. If there were a total of 25 handshakes (including all of Chris's), how many hands did Chris shake?

(A) 3 (B) 4 (C) 5 (D) 6 (E) 7

Solution

Suppose there are n guests at the party before Chris arrives. In order to form one handshake we must choose one guest from the n present and then we must choose another guest from the remaining $n-1$ present. There are n ways to choose the first guest and $n-1$ ways to choose the second, so there are $n \cdot (n-1)$ ways to choose the two guests in this order. Since shaking hands using the reverse order of the two guests produces the same handshake, we divide $n \cdot (n-1)$ by 2 to eliminate duplicates in the previous count. This gives a total of $\frac{n^2-n}{2}$ handshakes among the first n guests at the party. If Chris shakes hands with $m < n$ of the guests already present, then

$$\frac{n^2-n}{2} + m = 25$$

Now we try values of $n \geq 2$ until we find a value of m (with $m < n$) for which the equation is true. We quickly arrive at $n = 6$ and $m = 4$.

Alternate solution:

Let n be the number of guests before Chris arrives.

If $n = 2$ then there was just one handshake before her arrival.

If $n = 3$ then there were two additional handshakes before her arrival (for a total of $1 + 2 = 3$).

If $n = 4$ then there were 3 additional handshakes before her arrival (for a total of $1 + 2 + 3 = 6$).

Continuining in this way, we can make the following table:

n (number of guests)	2	3	4	5	6	7
handshakes before Chris arrived	1	3	6	10	15	21

We see that for $n \leq 6$ then there are not enough people for 25 handshakes (including Chris's). So Chris participated in $25 - 21 = 4$ handshakes.

The correct answer is (B).

Answer: B

10. Recall that $n! = n \cdot (n-1)(n-2) \cdots 2 \cdot 1$. For example, $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$. Which of the following numbers is a perfect square?

(A) $\frac{4!5!}{2}$ (B) $\frac{5!6!}{2}$ (C) $\frac{6!7!}{2}$ (D) $\frac{7!8!}{2}$ (E) $\frac{8!9!}{2}$

Solution
Each of the five choices can be placed in the form $\frac{[(n-1)!)]^2 \cdot n}{2}$ for some integer, n, so it remains to find the choice for which $\frac{n}{2}$ is a perfect square. We calculate $\frac{5}{2} = 24$ is not a perfect square; $\frac{6}{2} = 3$ is not a perfect square; $\frac{7}{2}$ is not a perfect square; but $\frac{8}{2} = 4$ is a perfect square.

The correct answer is (D).

Answer: D

11. Let $S = \{a, b, c, d, e\}$ be a set of positive integers with $a < b < c < d < e$. When two different numbers are chosen from this set, the possible values for their sum are:

$$165, 170, 175, 177, 182, 187, 190, 195, 200, 207.$$

Determine the value of c.

(A) 90 (B) 91 (C) 92 (D) 93 (E) 94

Solution
There are ten distinct sums, and each of the numbers a, b, c, d and e appears in exactly four of these sums. Adding all ten equations, we have

$$4a + 4b + 4c + 4d + 4e = 4(a + b + c + d + e) = 165 + \cdots + 207,$$

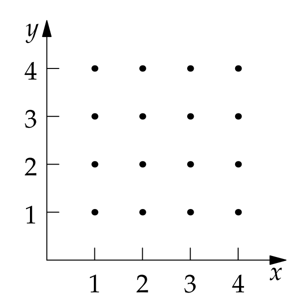
and this reduces to $a + b + c + d + e = 462$. The inequality relationship $a \leq b \leq c \leq d \leq e$ implies $a + b$ is the smallest sum and $d + e$ the largest, so $a + b = 165$ and $d + e = 207$. Now we can write

$$c = (a + b + c + d + e) - (a + b) - (d + e) = 462 - 165 - 207 = 90.$$

The correct answer is (A).

Answer: A

12. In the xy-plane, consider the sixteen points (x, y) with x and y both integers such that $1 \leq x \leq 4$ and $1 \leq y \leq 4$ (as shown in the diagram). Determine the number of triangles with positive area whose three vertices are chosen from these sixteen points.



(A) 496 (B) 516 (C) 520
(D) 528 (E) 560

Solution

Since a triangle with positive area is completely determined by any three non-collinear points, it is sufficient to count the number of ways of choosing three non-collinear points from the grid of sixteen points. The easiest way to count them is to use the so-called inclusion-exclusion principle: first we include all possible combinations of three points in the count; and then we exclude from the count those combinations which are collinear.

Reasoning in an elementary manner (or, if you know it, using the standard formula $C(n,k) = \frac{n!}{k!(n-k)!}$), we determine the number of ways of choosing three points from sixteen is $T = \frac{16 \cdot 15 \cdot 14}{3 \cdot 2 \cdot 1} = 560$. Next, we count the triples of collinear points in the grid. In the first place, we can find three collinear points in the grid by removing any one of the four points in any of the ten rows, columns, or main diagonals. This gives $4 \cdot 10 = 40$ such triples of collinear points. In second place, we can find four additional triples of collinear points by taking all three points in each of the four sub-diagonals in the grid. This gives a total of $40 + 4 = 44$ triples of collinear points and, hence, a total of $560 - 44 = 516$ triangles with positive area in the grid.

The correct answer is (B).

Answer: B