BRITISH COLUMBIA SECONDARY SCHOOL MATHEMATICS CONTEST, 2016

Senior Preliminary Problems & Solutions

1. Three hedgehogs (Ronny, Steph and Pat) are having a race against two tortoises (Ellery and Olly). At some point in the race Steph (S) is 10 m behind Olly (O), and Olly is 25 m ahead of Ronny (R). Ronny is 5 m behind Ellery (E), and Ellery is 25 m behind Pat (P). The order, from first to last, of the five racers at this point in the race spells the word:

   (A) PORES (B) POSER (C) PROSE (D) ROPES (E) SPORE

   Solution

   If we place Steph at the origin on the number line, then, using the given relative distances and the first initial of each contestant, we can form the list of ordered pairs

   $$(0, S), (10, O), (-15, R), (-10, E), (15, P).$$

   Next, using the size of the first coordinate, we arrange the pairs from left to right to obtain the revised list

   $$(-15, R), (-10, E), (0, S), (10, O), (15, P).$$

   Finally, reading the second coordinates in reverse order spells POSER.

   Answer: B

2. If it takes 864 digits to number the pages in a book, how many numbered pages are in the book?

   (A) 286 (B) 288 (C) 324 (D) 325 (E) 864

   Solution

   Let T be the number of pages in the book. Pages 1 to 9 require 1 digit each; pages 10 to 99 require 2 digits each; and pages 100 to T require 3 digits each. We have $9(1) + 90(2) + (T - 99)(3) = 864$, so $T = 324$.

   Answer: C

3. A two-digit integer is m times the sum of its digits. When the digits are reversed, the new number is n times the sum of its digits. What is the value of $m + n$?

   (A) 5 (B) 8 (C) 9 (D) 10 (E) 11

   Solution

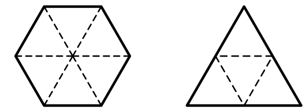
   Let a and b be the digits in the integer. Then $m(a + b) = 10a + b$ and $n(a + b) = 10b + a$. Adding these equations gives $(m + n)(a + b) = 11(a + b) \Rightarrow m + n = 11$.

   Answer: E

4. A wire is cut into two pieces of equal length. One is bent to form an equilateral triangle with area 2, and the other is bent to form a regular hexagon. What is the area of the hexagon?

   (A) 2 (B) $\frac{3}{2}\sqrt{3}$ (C) 3 (D) $2\sqrt{3}$ (E) 4

Solution



Since the two pieces of wire are of equal length the perimeters of the hexagon and triangle are equal, and it is apparent that the side length of the triangle is twice the side length of the hexagon. Hence, the triangle and hexagon can be divided into congruent equilateral triangles, as shown in the diagram. There are 6 triangles in the hexagon and 4 in the triangle. Therefore, if the area of the triangle is 4, then the area of the hexagon is 6. So, if the area of the triangle is 2, then the area of the hexagon is 3.

Alternative solution:

First note that straightforward application of properties of similar triangles to the standard formula for the area A of a triangle reveals that the area A of an equilateral triangle with side length S is $A = \frac{S^2 \cdot \sqrt{3}}{4}$.

If $2x$ is the length of the wire, then the side length of the equilateral triangle is $\frac{x}{3}$, so its area A_t is

$$A_t = \frac{(\frac{x}{3})^2 \cdot \sqrt{3}}{4} = \frac{x^2 \cdot \sqrt{3}}{36} = 2.$$

Solving for x^2, we have $x^2 = 24\sqrt{3}$. Now, the six small hexagons comprising the area of the given large hexagon each have side length $\frac{x}{6}$, so the area A_h of the large hexagon is

$$A_h = 6 \cdot \frac{(\frac{x}{6})^2 \cdot \sqrt{3}}{4} = 6 \cdot \frac{24\sqrt{3}}{(36)(4)} = 3.$$

Answer: C

5. Dale's lawn, which is circular with a diameter of 30 m, is in need of re-sodding. Dale can only buy sod in 50 cm wide strips. Which of the following best approximates the total length of sod (measured in metres) that Dale needs?

(A) 500 (B) 900 (C) 1500 (D) 3000 (E) 4500

Solution

Since we can fit 60 strips of width 50cm across the diameter of the circle, the strips cover the circular region quite well.

It follows that the area, A_c, of the circle can serve as a good upper bound on the total area A_s of the strips.

If we let L denote the total length of the strips, then $A_c = \pi r^2 = 225\pi \approx \frac{1}{2}L$.

Using 3 as a lower bound on π, we find 1350 is a reasonable lower bound on the total length of the strips.

Considering the rather crude estimate of π we used in the calculation, we conclude 1350 is a close enough underestimate of the choice 1500.

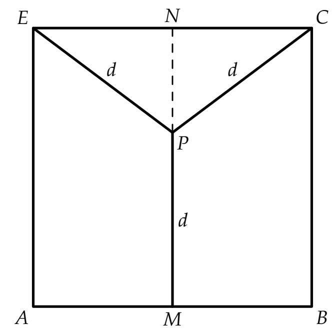
Answer: C

6. The point P is inside a square whose side length is 16. Its distance from two adjacent vertices and the midpoint of the side opposite these vertices has a common value d. What is the value of d?

(A) $16\left(2-\sqrt{2}\right)$ (B) 8 (C) $\dfrac{16}{\sqrt{2}}$ (D) $4\left(2+\sqrt{2}\right)$ (E) 10

Solution

Triangle CPD is an isosceles triangle and the segment NP, where N is the midpoint of side CD is perpendicular to CD. Further, line segment MP is perpendicular to side AB. Hence, $MP + NP = 16$. Using Pythagoras' theorem in triangle CNP gives



$$\sqrt{d^2 - 8^2} + d = 16 \Rightarrow \sqrt{d^2 - 64} = 16 - d$$

Squaring both sides gives

$$d^2 - 64 = (16 - d)^2 = 256 - 32d + d^2 \Rightarrow 32d = 320 \Rightarrow d = 10$$

Alternative solution:

Let S denote the length in question and let $X = 16 - S$. We have $X + S = 16$ and

$$S^2 = X^2 + 64 = (16 - S)^2 + 64 = 16^2 - 32S + S^2 - 64,$$

so $S = 10$.

Answer: E

7. Consider the product

$$(0.\overline{9})\,(0.\overline{6})\,(0.24\overline{9})\,(0.1\overline{9})\,(0.8\overline{3}).$$

The line over a digit means that the digit is repeated indefinitely. For example,

$$0.\overline{3} = 0.33333333\cdots = \frac{1}{3} \text{ and } 0.1\overline{6} = 0.16666666\cdots = \frac{1}{6}.$$

Which of the following is equal to the product above?

(A) $\frac{5}{24}$ (B) $\frac{1}{24}$ (C) $\frac{1}{30}$ (D) $\frac{1}{36}$ (E) $\frac{1}{48}$

Solution

Note that

$$0.\overline{9} = 0.999999\ldots = 1, 0.\overline{6} = 0.666666\ldots = \tfrac{2}{3}, 0.24\overline{9} = 0.249999\ldots = 0.25 = \tfrac{1}{4},$$
$$0.1\overline{9} = 0.199999\ldots = 0.2 = \tfrac{1}{5}, \text{ and } 0.8\overline{3} = 0.833333\ldots = \tfrac{5}{6}$$

So that

$$(0.\overline{9})\,(0.\overline{6})\,(0.24\overline{9})\,(0.1\overline{9})\,(0.8\overline{3}) = 1 \times \frac{2}{3} \times \frac{1}{4} \times \frac{1}{5} \times \frac{5}{6} = \frac{1}{36}$$

Answer: D

8. Find the sum of all values x such that $(x^2 - 7x + 11)^{x^2+3x-10} = 1$.

(A) 0 (B) 2 (C) 4 (D) 7 (E) 9

Solution

Let $b = x^2 - 7x + 11$ and $m = x^2 + 3x - 10$. We want to solve the equation $b^m = 1$ for all real values x. There are a number of cases to consider. First note the form is indeterminate if both b and m are zero, so we avoid this case.

Case 1: $b = 1$ and m is real.

$$\begin{aligned} b = x^2 - 7x + 11 = 1 &\Leftrightarrow x^2 - 7x + 10 = 0 \\ &\Leftrightarrow (x-2)(x-5) = 0 \\ &\Leftrightarrow x = 2 \text{ or } x = 5. \end{aligned}$$

Now, if $x = 2$ then $m = 2^2 + 3(2) - 10 = 0$ is real. If $x = 5$, then $m = 5^2 + 3(5) - 10 = 30$ is also real. So both $x = 2$ and $x = 5$ are solutions.

Case 2: $m = 0$.

$$\begin{aligned} m = x^2 + 3x - 10 = 0 &\Leftrightarrow (x+5)(x-2) = 0 \\ &\Leftrightarrow x = -5 \text{ or } x = 2. \end{aligned}$$

so $x = -5$ is a new solution.

Case 3: $b = -1$ and m is even.

$$\begin{aligned} b = x^2 - 7x + 11 = -1 &\Leftrightarrow x^2 - 7x + 12 = 0 \\ &\Leftrightarrow (x-3)(x-4) = 0 \\ &\Leftrightarrow x = 3 \text{ or } x = 4. \end{aligned}$$

For both of these values m is even, so we have two new solutions, $x = 3$ and $x = 4$.

The sum of the solutions is $2 + 5 + (-5) + 3 + 4 = 9$.

Answer: E

9. Let x be a real number between 0 and 1. When a discount x is applied to the price P of an item in a store, the price of the item is reduced by xP dollars. Three successive discounts of $(\frac{1}{3}x)$ results in a price that is the same as if there were one discount of:

(A) x (B) $x - \dfrac{x^2}{3} + \dfrac{x^3}{27}$ (C) $x + \dfrac{x^2}{3} - \dfrac{x^3}{27}$ (D) $x - \dfrac{x^3}{27}$ (E) $x + \dfrac{x^3}{27}$

Solution

After three successive discounts of $\frac{1}{3}x$ the new price is

$$P - \left(\tfrac{1}{3}x\right)P - \left[P - \left(\tfrac{1}{3}x\right)P\right]\left(\frac{1}{3}x\right) - \left\{P - \left(\tfrac{1}{3}x\right)P - \left[P - \left(\tfrac{1}{3}x\right)P\right]\left(\frac{1}{3}x\right)\right\}\left(\frac{1}{3}x\right) = P\left(1 - \tfrac{1}{3}x\right)^3$$

Further

$$P\left(1 - \tfrac{1}{2}x\right)^3 = P\left(1 - x + \tfrac{1}{3}x^2 - \tfrac{1}{27}x^3\right)$$

So reduction in price is

$$P - P\left(1 - x + \tfrac{1}{3}x^2 - \tfrac{1}{27}x^3\right) = P\left(x - \tfrac{1}{3}x^2 + \tfrac{1}{27}x^3\right)$$

and the discount is $x - \frac{1}{3}x^2 + \frac{1}{27}x^3$.

Answer: B

10. Nine people attend a dinner where there are three choices for the type of meal. Three people order Combo A, three order Combo B, and three order Combo C. The server distributes the nine meals in random order. In how many different ways can exactly one person receive the correct meal?

(A) 54 (B) 135 (C) 162 (D) 216 (E) 270

Solution

The table below gives the main structure of the solution:

Person:	1	2	3	4	5	6	7	8	9
Order:	A	A	A	B	B	B	C	C	C
Served:	A	C	C	A	A	C	B	B	B
	A	B	C	A	C	C	A	B	B
	A	C	B	A	C	C	A	B	B

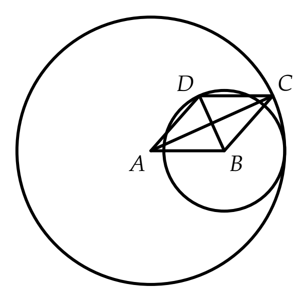
Suppose that the first person gets the correct meal A. Then the other two of that meal go either to the same group (line 3 in table above) or different groups (lines 4 and 5 in the table). In the first case there is only one way to distribute the C meals which in turn determines how the B meals are distributed. In the second case two of the C meals must go to the B group (no choice) and one to the A group (2 choices).
Now start counting:

- Choose the person who receives the correct meal (9 ways).
  Now either
- the other two meals go to the same group. Choose that group (2 ways) and then choose the people (3 ways). There are no other choices.
  or
- the other two meals go to different groups. Choose the person in the second group (3 ways) and the person in the third group (3 ways) who receive them. Now we have to determine how the meals are distributed to the remaining members of the A group (2 ways).

This gives $9 \times (2 \times 3 + 3 \times 3 \times 2) = 216$ ways.

Answer: D

11. A rhombus, $ABCD$, has sides of length 1. A circle with centre A passes through C (the opposite vertex.) Likewise, a circle with centre B passes through D. If the two circles are tangent to each other, find the area of the rhombus.



(A) $\frac{1}{2}$ (B) $\frac{2}{3}$ (C) $\frac{5}{8}$

(D) $\frac{3}{4}$ (E) 1

Solution

If the circles are tangent then the point of tangency must be on the line which joins their centres. Let R be the radius of the large circle (the length of the longer diagonal) and r be the radius of the small circle (the length of the shorter diagonal). Then $R = 1 + r$. The diagonals bisect each other at right angles so

$$\frac{R^2}{4} + \frac{r^2}{4} = 1 \Rightarrow R^2 + r^2 = 4 \Rightarrow 2r^2 + 2r - 3 = 0 \Rightarrow r = \frac{-2+\sqrt{28}}{4} \text{ and } R = 1 + r = \frac{2+\sqrt{28}}{4}$$

So the area of the rhombus is

$$\frac{rR}{2} = \frac{1}{2}\left(\frac{-2+\sqrt{28}}{4}\right)\left(\frac{2+\sqrt{28}}{4}\right) = \frac{-4+28}{32} = \frac{3}{4}$$

Alternative solution:

Let E be the point where the extension of AB intersects the circle. Since the circles are tangent, AE is a radius.

Let R and r denote respectively the radii of the larger and smaller circle. We have $R = AC = AB + BE = 1 + r$.

Since $ABCD$ is a rhombus, $(\frac{R}{2})^2 + (\frac{r}{2})^2 = 1^2 = 1$, so $R^2 + r^2 = 4$.

Substituting $R = r + 1$ into this equation, solving for r, and rejecting the extraneous root, we have $(r+1)^2 + r^2 = 4$, so $r = \frac{-1+\sqrt{7}}{2}$ and $R = \frac{1+\sqrt{7}}{2}$.

Evidently, the area of the rhombus is given by $\frac{R \cdot r}{2} = \frac{3}{4}$.

Answer: D

12. For $n = 1, 2, 3, \ldots$, the n^{th} *harmonic number* H_n is defined by

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

Which of the following is a formula for the sum $H_1 + H_2 + \cdots + H_n$?

(A) $(n+1)\,H_n - n$ (B) $(n-1)\,(H_n + 1)$ (C) $nH_n + \dfrac{1}{n}$

(D) $(n+1)\,H_n + n - 2$ (E) $(2n-1)\,H_n - n$

Solution

Writing out the sum

$$\begin{aligned}
S_n = H_1 + H_2 + \cdots + H_n \quad = \quad & 1 \quad + \\
& 1 \quad + \frac{1}{2} + \\
& 1 \quad + \frac{1}{2} + \frac{1}{3} + \\
& 1 \quad + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \\
& \quad\ \vdots \\
& 1 \quad + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{n}
\end{aligned}$$

Adding the terms above in columns gives

$$\begin{aligned}
S_n &= n + \frac{n-1}{2} + \frac{n-2}{3} + \cdots + \frac{n-n+1}{n} \\
&= n + \frac{n}{2} + \frac{n}{3} + \cdots + \frac{n}{n} - \left(\frac{2-1}{2} + \frac{3-1}{3} + \cdots + \frac{n-1}{n} \right) \\
&= n \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \right) - \left(1 - \frac{1}{2} + 1 - \frac{1}{3} + \cdots 1 - \frac{1}{n} \right) \\
&= nH_n - [n - 1 - (H_n - 1)] = nH_n - (n - 1 - H_n + 1) \\
&= nH_n + H_n - n = (n+1)\,H_n - n
\end{aligned}$$

Answer: A