BRITISH COLUMBIA SECONDARY SCHOOL MATHEMATICS CONTEST, 2016
Junior Preliminary Problems & Solutions

1. Which of the following is *not* a factor of 48?

   (A) 6   (B) 12   (C) 16   (D) 18   (E) 24

   Solution

   If we divide, successively, each of the five listed choices into 48 (using the division algorithm, say), we find that each of 6, 12, 16 and 24 are factors of 48, but the fractional part of $\frac{48}{18} = \frac{8}{3}$ is non-zero, so 18 is not a factor of 48.

   Alternativelym any number that is not a factor of 48 must have a prime factor that does not appear in the factorization $48 = 3 \cdot 2^4$. Since that factor 3 appears twice in $18 = 2 \cdot 3^2$, 18 is not a factor of 48.

   | Answer: D |

2. A 3×3 magic square consists of nine different numbers placed in a grid in such a way that the sum of the numbers in each row, each column, and the two main diagonals is the same. The numbers 5, 10, 15, 20, 25, 30, 35, 40, and 45 are used to form a magic square. Some of these numbers have been placed, as shown. Find the the value of X.

20	45	a
		X
		30

   (A) 10   (B) 15   (C) 35   (D) 40   (E) 45

   Solution

   The sum of the numbers in the entire square is $5 + 10 + 15 + \cdots + 45 = 225$. This is also the sum of the three rows and the three columns. Hence, the sum of each row and column is $\frac{225}{3} = 75$. The number in the upper right hand corner is $75 - 20 - 45 = 10$, so $X = 75 - 10 - 30 = 35$.

   Alternative solution:

   The sum of the first row is $65 + a$ and the sum of the third column is $a + X + 30$. Since these are equal

   $$65 + a = a + X + 30 \Rightarrow X = 35$$

   | Answer: C |

3. Maddie and her mother's ages are both perfect squares. In four years, her mother will be five times her age. What is the sum of their ages now?

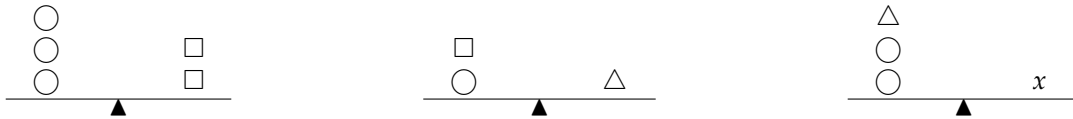   (A) 26   (B) 29   (C) 37   (D) 40   (E) 45

   Solution

   If we denote Maddie's current age by A and her mother's current age by M, then $M + 4 = 5(A + 4)$, so

$M = 5A + 16$. Now we successively substitute the perfect squares $1, 4, 9...$ for A until we find that M is a perfect square. We quickly find $A = 4$ and $M = 36$ satisfy the required conditions so $M + A = 40$.

Answer: D

4. In the diagram below all scales are balanced. How many □'s does x represent?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Solution

We set-up a system of three linear equations in three variables. In place of the symbols shown, we use the variables C, S, and T. Presumably, they represent the masses of the three different shapes. We infer from the figure that

$$3C - 2S = 0, C + S - T = 0, \text{ and } 2C - xS + T = 0.$$

Add the second and third equations to obtain the fourth equation $3C + (1 - x)S = 0$. Now subtract the first equation from the fourth to obtain $(1 - x)S + 2S = 0$. Since the choice $S = 0$ renders the problem vacuous, it now follows $x = 3$.

Answer: C

5. Three hedgehogs (Ronny, Steph and Pat) are having a race against two tortoises (Ellery and Olly). At some point in the race Steph (S) is 10 m behind Olly (O), and Olly is 25 m ahead of Ronny (R). Ronny is 5 m behind Ellery (E), and Ellery is 25 m behind Pat (P). The order, from first to last, of the five racers at this point in the race spells the word:

(A) PORES (B) POSER (C) PROSE (D) ROPES (E) SPORE

Solution

If we place Steph at the origin on the number line, then, using the given relative distances and the first initial of each contestant, we can form the list of ordered pairs

$$(0, S), (10, O), (-15, R), (-10, E), (15, P).$$

Next, using the size of the first coordinate, we arrange the pairs from left to right to obtain the revised list

$$(-15, R), (-10, E), (0, S), (10, O), (15, P).$$

Finally, reading the second coordinates in reverse order spells POSER.

Answer: B

6. One sunny afternoon, there are a lot of people walking their dogs in a dog park. In total there are 40 heads and 106 legs. What is the number of people minus the number of dogs?
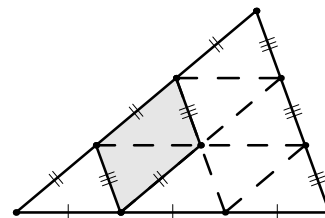
(A) 2 (B) 6 (C) 8 (D) 13 (E) 14

Solution

If h denotes the number of humans and d denotes the number of dogs, then we are given $h + d = 40$ and $2h + 4d = 106$. From the first equation we have $d = 40 - h$. Substituting d into the second equation gives $2h + 4(40 - h) = 106$ so $h = 27$, $d = 40 - 27 = 13$, and, finally, $h - d = 27 - 13 = 14$.

Answer: E

7.   The shaded area is what fraction of the total area of the triangle?

(A) $\frac{1}{5}$ (B) $\frac{2}{9}$ (C) $\frac{1}{4}$

(D) $\frac{1}{3}$ (E) $\frac{3}{8}$

Solution

See diagram. The total area can be subdivided into 9 congruent triangles, 2 of which make up the shaded area. The fraction is therefore $\frac{2}{9}$.

Answer: B

8.   There is \$150 in a store's cash register at the beginning of a day, and \$348 at the end. If a 10% sales tax is charged on each purchase, how much of the money collected was sales tax?

(A)   \$18   (B)   \$19   (C)   \$20   (D)   \$21   (E)   \$22

Solution

If we denote the amount of tax by x, then we are given $10x + x = 349 - 150 = 198$, so $x = 18$.

If we begin, instead, with x as the value of the merchandise sold that day, then we have $x + \frac{x}{10} = 349 - 150 = 198$, so $x = 180$ and the tax is $\frac{x}{10} = 18$.

Answer: A

9.   A baseball league has 9 teams. During the season, each team plays every other team 4 times. How many games are played in total?

(A)   36   (B)   72   (C)   81   (D)   144   (E)   288

Solution

First consider the simpler problem in which each team plays every other team exactly once. In this case, each team needs to play 8 games. If we write a list of all teams matched against team #1, then and another list of all teams matched with team #2, etc, we will have listed $9 \times 8 = 72$ pairings. But each unique pairing will have appeared twice (e.g. the game pairing team #1 with team #2 will appear once in the list of games for team #1, and again in the list of games for team #2). Thus the number of distinct pairings is $72/2 = 36$.

In the whole season each of these pairings occurs 4 times, so there are $4 \times 36 = 144$ games.

Answer: D

10.   The value of the expression

$$\frac{\left(\dfrac{1}{2} - \dfrac{2}{3}\right)}{\left(\dfrac{3}{4} - \dfrac{4}{5}\right)} \div \frac{5}{6}$$

is:

(A) $\frac{1}{144}$ (B) $\frac{1}{100}$ (C) $\frac{5}{6}$ (D) $\frac{25}{9}$ (E) 4

Solution

There are many ways to carry-out the calculation, some more efficient than others. One possible calculation - which exploits the distributive and cancelation laws - is

$$\frac{\left(\dfrac{1}{2}-\dfrac{2}{3}\right)}{\left(\dfrac{3}{4}-\dfrac{4}{5}\right)} \div \frac{5}{6} = \frac{\left(\dfrac{1}{2}-\dfrac{2}{3}\right)}{\left(\dfrac{3}{4}-\dfrac{4}{5}\right)} \times \frac{6}{5} = \frac{\dfrac{3-4}{15}}{\dfrac{-1}{4}-4} = \frac{-1}{-1} = 4$$

Yet another solution:

Inverting $\frac{5}{6}$, changing the operation to multiplication, and multiplying the given quotient by $\frac{60}{60}$ we obtain $\frac{\frac{1}{2}-\frac{2}{3}}{\frac{3}{4}-\frac{4}{5}} \cdot \frac{60}{60} \cdot \frac{6}{5} = \frac{30-40}{45-48} \cdot \frac{6}{5} = \frac{-10}{-3} \cdot \frac{6}{5} = 4.$

Another solution:

More generally, if n is any number for which all of the quotients in the expression below are defined, then straight-forward algebraic manipulation gives

$$\frac{\dfrac{n}{n+1}-\dfrac{n+1}{n+2}}{\dfrac{n+2}{n+3}-\dfrac{n+3}{n+4}} \div \frac{n+4}{n+5} = \frac{(n+3)(n+5)}{(n+1)(n+2)}$$
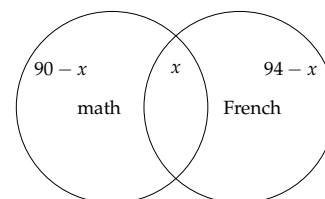
Setting $n = 1$ gives the same value of 4.

Answer: E

11. At a particular high school, 90% of the students take math, 94% take French, and x% take both math and French. Find the minimum value of x.

(A) 0%   (B) 74%   (C) 84%   (D) 90%   (E) 96%

Solution

A Venn diagram helps visualize the problem. This diagram contains at most 100% and at least 94% of all students. Thus

$$94 \le (90 - x) + x + (94 - x) \le 100$$
$$\implies 94 \le 184 - x \le 100$$
$$\implies -90 \le -x \le -84$$
$$\implies 90 \ge x \ge 84.$$

So the minimum value of x is 84%.

Answer: C

12. A wire is cut into two pieces of equal length. One is bent to form an equilateral triangle with area 2, and the other is bent to form a regular hexagon. What is the area of the hexagon?

(A) 2   (B) $\frac{3}{2}\sqrt{3}$ (C) 3   (D) $2\sqrt{3}$ (E) 4

Solution

Since the two pieces of wire are of equal length the perimeters of the hexagon and triangle are equal, and it is apparent that the side length of the triangle is twice the side length of the hexagon. Hence, the triangle and hexagon can be divided into congruent equilateral triangles, as shown in the diagram. There are 6 triangles in the hexagon and 4 in the triangle. Therefore, if the area of the triangle is 4, then the area of the hexagon is 6. So, if the area of the triangle is 2, then the area of the hexagon is 3.
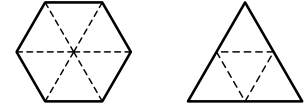
Alternative solution:

First note that straightforward application of properties of similar triangles to the standard formula for the area A of a triangle reveals that the area A of an equilateral triangle with side length S is $A = \frac{S^2 \cdot \sqrt{3}}{4}$.

If $2x$ is the length of the wire, then the side length of the equilateral triangle is $\frac{x}{3}$, so its area A_t is

$$A_t = \frac{(\frac{x}{3})^2 \cdot \sqrt{3}}{4} = \frac{x^2 \cdot \sqrt{3}}{36} = 2.$$

Solving for x^2, we have $x^2 = 24\sqrt{3}$. Now, the six small hexagons comprising the area of the given large hexagon each have side length $\frac{x}{6}$, so the area A_h of the large hexagon is

$$A_h = 6 \cdot \frac{(\frac{x}{6})^2 \cdot \sqrt{3}}{4} = 6 \cdot \frac{24\sqrt{3}}{(36)(4)} = 3.$$

Answer: C