BRITISH COLUMBIA SECONDARY SCHOOL MATHEMATICS CONTEST, 2015

Solutions

Junior Preliminary

1. If we denote Raven Riddle's age by n, then we are given $2(n-8) = n+4$, from which we find $n = 20$.

Answer is (C).

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2. Each of the given polygonal regions can be partitioned into some number m of squares each of area 1, and some number n of triangular regions each of area $\frac{1}{2}$, so a formula for the area of a typical region is $A = m + \frac{1}{2}n$. Counting m and n for each region and substituting them into the formula, we find the areas are, in order:

$$\begin{array}{llcllll} \text{A:} & 4\,(1)+2\left(\frac{1}{2}\right) & = & 5 & \text{B:} & 4\,(1)+2\left(\frac{1}{2}\right) & = 5 \\ \text{C:} & 4\,(1)+3\left(\frac{1}{2}\right) & = & 5.5 & \text{D:} & 3\,(1)+3\left(\frac{1}{2}\right) & = 4.5 \\ \text{E:} & 2\,(1)+6\left(\frac{1}{2}\right) & = & 5 & & & \end{array}$$

We conclude the largest area is that of the third listed polygon, labeled (C).

Answer is (C).

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3. Evidently, the second grid can be obtained from the first grid by a quarter rotation clockwise followed by a reflection in the vertical axis. We must perform the inverses of these operations - a quarter rotation counterclockwise and a reflection in the vertical axis - on the third grid. Since reflections do not, in general, commute with rotations, we must be careful to reverse the order of the inverse operations. After performing the required transformations on the third grid, we conclude the original position and orientation of the letter L is that given in the fourth choice.

Answer is (D).

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4. The first eleven prime numbers are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, and 31. The sum of these numbers is

$$2+3+5+7+11+13+17+19+23+29+31 = 160$$

Answer is (D).

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5. Taking common denominators, inverting and multiplying, we have

$$2015 = \frac{\dfrac{1}{w}+\dfrac{1}{z}}{\dfrac{1}{w}-\dfrac{1}{z}} = \frac{\dfrac{z+w}{wz}}{\dfrac{z-w}{wz}} = \frac{z+w}{z-w} = -\frac{w+z}{w-z}$$

so $\dfrac{w+z}{w-z} = -2015$.

Answer is (A).

6. Since, by assumption, triangle ABC is a right triangle, and

$$AB^2 + BC^2 = 9^2 + 12^2 = 81 + 144 = 225 = 15^2$$

we see $AC = 15$ by Pythagoras' theorem. Now,

$$AD^2 + AC^2 = 8^2 + 15^2 = 64 + 225 = 289 = 17^2$$

so Pythagoras' theorem implies that triangle DAC is a right triangle. It then follows that the area of quadrilateral $ABCD$ is

$$\frac{1}{2}(8 \times 15) + \frac{1}{2}(12 \times 9) = 60 + 54 = 114$$

Answer is (E).

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7. A is sumfree because the sum of two odd numbers is even. B is not sumfree (quite the opposite, in fact) because the sum of two even numbers is always even. The set of primes is not sumfree because $2 + 3 = 5$. The set of squares is not sumfree because $3^2 + 4^2 = 5^2$. E is not sumfree since, for example, $2^2 + 2^2 = 2^3$. F is sumfree since, if $m \leq n$, then $3^m + 3^n = 3^m (3^{n-m} + 1)$. Any power of 3 is odd, so that $3^{n-m} + 1$ must be even, and so cannot be a power of 3. Therefore, two of the sets above are sumfree.

Answer is (C).

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8. Since Alex must take out 44 eggs to be certain of getting at least one white egg, there must be 43 eggs that are blue or red. Since 4 of them are blue, 39 of them must be red. It follows that Alex must take out 40 eggs to be sure of getting at least one white or one blue egg.

Answer is (B).

9. Let x and y denote the lengths of the sides of the shaded region. Since ratios of corresponding sides of similar triangles are equal, we have

$$\frac{5}{10} = \frac{x}{2} = \frac{y}{5}$$

from which it follows $x = 1$ and $y = \frac{5}{2}$. The area of the shaded region is the difference between the areas of two triangles: one with height $y = \frac{5}{2}$ and base 5 and the other with height $x = 1$ and base 2. This gives

$$\text{area of shaded region} = \frac{1}{2}(5)\left(\frac{5}{2}\right) - \frac{1}{2}(1)(2) = \frac{25}{4} - 1 = \frac{21}{4}$$

Alternative solution:

The area of the shaded region is the area of a trapezoid with the two parallel sides with lengths $x = 1$ and $y = \frac{5}{2}$ and height 3. Hence,

$$\text{area of shaded region} = \frac{3}{2}\left(1 + \frac{5}{2}\right) = \frac{3}{2} \times \frac{7}{2} = \frac{21}{4}$$

Answer is (A).

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10. Label the colours G, Y, R, B, and W. We provide a process for placing the houses so that all possible arrangements are considered, and none of them are considered more than once. Begin by placing W and Y with W first. We can now place G relative to W and Y in three ways. Now we have the colours G, W, and Y in some order. Next, place R in one of the three possible positions, and then place B (after and not together with R). In the first case there are three possibilities; in the second case there are two possibilities, and in the third case there is one possibility. Thus, by the multiplication and addition principles, there are $3 \cdot (3 + 2 + 1) = 18$ possible orderings of these five houses on Bernard Street.

Answer is (B).

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11. Runner B goes $24 - 6 = 18$ km in the time it takes runner C to go $24 - 9 = 15$ km. Let x be the distance runner C goes in the time it takes runner B to go the 6 km required to finish the race. Then

$$\frac{18}{15} = \frac{6}{x} \Rightarrow x = \frac{15 \times 6}{18} = 5$$

Hence, the number of kilometres runner C will have to run to complete the race from the time when runner B crosses the finish line is $9 - 5 = 4$.

Alternative solution:

Let V_A, V_B, and V_C denote the speeds of runners A, B, and C, respectively, and suppose it takes 1 unit of time t for A to complete the race. Then $V_A = 24$, $V_B = 18$, and $V_C = 15$. It will take

$$\text{time} = \frac{\text{distance}}{\text{speed}} = \frac{6}{18} = \frac{1}{3}$$

units of time for B to finish, during which time C will have moved

$$\text{distance} = \text{time} \times \text{velocity} = \frac{1}{3} \times 15 = 5\,\text{km}$$

In total, C has then gone $15 + 5 = 20$ km and thus has $24 - 20 = 4$ km to complete.

Answer is (D).

12. Positive real numbers $a \leq b \leq c$ are the lengths of the sides of a non-degenerate triangle if and only if the numbers satisfy the strict *triangle inequality* $a + b > c$. We must choose a, b, and c from the set $S = \{1, 2, 3, 4\}$ so that these inequalities are satisfied. First we reject from the $4^3 = 64$ possible ordered triples taken from S all of those that do not satisfy the inequality $a \leq b \leq c$. Next we reject from those remaining those that do not satisfy the condition $a + b > c$. Finally, we reject from the remaining triangle those that are duplicated by congruence (of which there are none) and similarity. Note that the triangles corresponding to any two of the given triples are similar if and only if the given triples are multiples of each other, we are left with the nine ordered triples $(1, 1, 1)$, $(1, 2, 2)$, $(1, 3, 3)$, $(1, 4, 4)$, $(2, 2, 3)$, $(2, 3, 3)$, $(2, 3, 4)$, $(3, 3, 4)$, and $(3, 4, 4)$, for a total of nine triangles.

Answer is (B).

Senior Preliminary

1. The laws of exponents give $V = (3^n)^3 = 3^{3n} = 9^{3^3} = (3^2)^{3^3} = 3^{54}$ so $3n = 54$ and $n = 18$.

Answer is (D).

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2. Informally, since the circumference of a circle is an increasing linear function of its radius and the area of a circle is an increasing quadratic function of its radius, we see the radius is an increasing linear function of the circumference, and, hence, the area of a circle is an increasing quadratic function of its circumference. It follows that the graph must look like the first choice shown.
More formally, if R, C, and A denote the radius, circumference, and area of some fixed circle, then $C = 2\pi R$ and $A = \pi R^2$, so $R = \frac{C}{2\pi}$ and, by substitution, $A = \pi[\frac{C}{2\pi}]^2 = \frac{C^2}{4\pi}$. We conclude, as above, that A is an increasing quadratic function of C.

Answer is (A).

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3. Note that $n!\,(n+1)! = (n!)^2\,(n+1)$. Since $(n!)^2$ contains only even powers, we see that

$$\frac{n!\,(n+1)!}{3} = \frac{(n!)^2\,(n+1)}{3}$$

can be a perfect square only if $\dfrac{n+1}{3}$ is a perfect square. In the expressions above

$$n + 1 = 24,\ 25,\ 26,\ 27,\ \text{and},\ 28$$

The only expression for which $\dfrac{n+1}{3}$ is a perfect square is $n + 1 = 27$, since $\dfrac{27}{3} = 9 = 3^2$.

Answer is (D).

4. Taking note of the coordinates of the given points, we see triangle OCD has a base of length 7 and a corresponding height of 2 so its area is $A = \frac{1}{2} \cdot 7 \cdot 2 = 7$. Similarly, triangle OAC has a base of length 5 and a height of 2 so its area is $A = \frac{1}{2} \cdot 5 \cdot 2 = 5$, and triangle ABC has a base of length 5 and a height of 2 so its area is $A = \frac{1}{2} \cdot 5 \cdot 2 = 5$. The area of pentagon $OABCD$ is the sum of the areas of the three triangles; that is, $7 + 5 + 5 = 17$.

Answer is (E).

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5. Applying the binomial theorem (or expanding the cube using long-multiplication), we compute

$$\begin{aligned} 5^3 = \left(x + \frac{1}{x}\right)^3 &= x^3 + 3x^2\left(\frac{1}{x}\right) + 3x\left(\frac{1}{x^2}\right) + \frac{1}{x^3} \\ &= x^3 + 3x + \frac{3}{x} + \frac{1}{x^3} = x^3 + 3\left(x + \frac{1}{x}\right) + \frac{1}{x^3} \\ &= x^3 + \frac{1}{x^3} + 3\,(5) = x^3 + \frac{1}{x^3} + 15 \end{aligned}$$

Therefore

$$x^3 + \frac{1}{x^3} = 5^3 - 15 = 110$$

Answer is (A).

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6. Let t_{11} denote the 11^{th} term of the sequence, and let s_n denote the sum of the first n terms. Substituting $n = 11$ and $n = 10$ into the given formula for s_n, we compute

$$t_{11} = s_{11} - s_{10} = [(11)\,(10)\,(2\,(11) + 3)] - [(10)\,(9)\,(2\,(10) + 3)] = 10\,(275 - 207) = 680$$

Answer is (C).

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7. First observe that

$$\frac{P}{Q} - \frac{Q}{P} = \frac{P^2 - Q^2}{P \cdot Q} = \frac{(P - Q)\,(P + Q)}{P \cdot Q}$$

Since $P > 0$ and $Q > 0$, we see that

$$\frac{P}{Q} - \frac{Q}{P} = \frac{(P - Q)\,(P + Q)}{P \cdot Q} = \frac{P + Q}{P \cdot Q} \Leftrightarrow P - Q = 1 \Leftrightarrow P = Q + 1$$

Evidently P must range over the values from 2 to 9 inclusive, as Q ranges over the values 1 to 8 inclusive. Hence, there are eight solutions; namely, $(1,2)$, $(2,3)$, $(3,4)$, $(4,5)$, $(5,6)$, $(6,7)$, $(7,8)$, and $(8,9)$.

Answer is (B).

8. Careful examination of the diagram reveals that the triangles in the diagram are in one-to-one correspondence with the pairs of intersection points lying on the five lines joining the vertical and horizontal lines in the diagram. Since there must be equal numbers of each, we count the pairs of points instead of the triangles. The lowest line has 6 intersection points. There are 6 ways to choose the first point and 5 ways to choose the second for a total of 30. Since the order of the points does not matter, we divide this by 2 to obtain $\frac{30}{2} = 15$ pairs. Counting similarly for the other lines, we find there are a total of $15 + 10 + 6 + 3 + 1 = 35$ triangles in the diagram.

Alternative solution:

Each triangle is determined uniquely by the intersection of three of the seven lines. The number of ways to choose three lines from the seven is

$$\binom{7}{3} = \frac{7!}{3! \cdot 4!} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35$$

Answer is (B).

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9. There are six distinct isosceles triangles that contain exactly two edges of the hexagon (one at each vertex of the hexagon), none that contain exactly one edge, and two that contain no edges (the equilateral triangles formed by choosing from the hexagon any three vertices no two of which are adjacent). Hence, there are eight distinct triangles with at least two equal sides in the regular hexagon.

Answer is (C).

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10. Since the exterior angles of a regular polygon sum to 360° degrees and there are six exterior angles on a hexagon, the exterior angle of the hexagon is $\dfrac{360°}{6} = 60°$. Also, the interior and exterior angles of a regular polygon must sum to 180°, so the interior angle of a hexagon must be 120°. Let P be the midpoint of the side BC, let E be the midpoint of BP, and let D be the upper right vertex of the internal hexagon. Since $\angle BDP = 120°$ we have $\angle BDE = \angle PDE = 60°$, so that BED and PED are 30°-60°-90° triangles with hypotenuse 1. Hence, $ED = \frac{1}{2}$ and $BE = EP = \frac{1}{2}\sqrt{3}$. Thus,

$$BC = 4\left(\tfrac{1}{2}\sqrt{3}\right) = 2\sqrt{3}$$

Further

$$AP = 1 + \tfrac{1}{2} + 1 + \tfrac{1}{2} = 3$$

Therefore,

$$\text{area triangle } ABC = \tfrac{1}{2}(3)\left(2\sqrt{3}\right) = 3\sqrt{3}$$

Answer is (E).

11. Let T, s, and f be total time (in hours) for the trip, Donny's starting position (in kilometres), and her final position (in kilometres), respectively, where T is a positive integer. We are given that

$$s = 100a + 10b + c, f = 100c + 10b + a$$
$$a \geq 1 \text{ and } a + b + c \leq 7$$

Then computing the distance traveled gives

$$f - s = 100c + 10b + a - (100a + 10b + c) = 99(c - a) \Rightarrow 55T = 99(c - a)$$

Hence, $c - a$ is multiple of 5, that is $c - a = 5k$ for some integer k. Since a and c are digits between 0 and 9, the only possible value of k is $k = 1$. Thus, $c = 5 + a$, and the restriction on the sum $a + b + c$ then gives $2a + b \leq 2$. Since $a \geq 1$, the only possible solution is $a = 1$ and $b = 0$, which gives $c = 5 + a = 6$. In this case $a^2 + b^2 + c^2 = 1^2 + 0^2 + 6^2 = 37$.

Answer is (D).

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12. Label the points $A = (a, 1)$, $B = (b, 3)$, $C = (c, 8)$, and $D = (d, 6)$. Let O be the intersection point of the perpendicular from D to the horizontal axis with the perpendicular from C to the vertical axis. Consider the triangle COD. The given square $ABCD$ consists of four such congruent triangles constructed in the same way from the other three corners, and a small square in the middle. By symmetry, the given numerical information implies the side length of the small square is 3. It follows the area of the given square is $A = 4\left(\frac{1}{2}\right)(2)(5) + 3^2 = 20 + 9 = 29$.

Answer is (B).

Junior Final, Part A

1. First note that $8 = 2^3$, $12 = 2^2 \times 3$, and $18 = 2 \times 3^2$. The smallest number divisible by all three is the least common multiple of these three numbers. Therefore, the number is $2^3 \times 3^2 = 72$, which is between 60 and 79.

Answer is (D).

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2. Let $\alpha = \angle ABC$ and $\beta = \angle BDA$. Since $AB = AC$

$$\angle ACB = \angle ABC = \alpha$$

Further, $DA = DB$ so that $\angle DAB = \angle DBA = \alpha$ and

$$2\alpha + \beta = 180 \Rightarrow \beta = 180 - 2\alpha$$

Further, since β is external to angle $\angle ADC$ it is given by

$$\beta = \alpha + 27$$

Equating the two expression for β and solving for angle α gives

$$3\alpha = 180 - 27 = 153 \Rightarrow \alpha = 51$$

Therefore, $\beta = \alpha + 27 = 78$.

Answer is (B).

3. The pattern repeats itself every 6 seconds. Now, $2015 = 6 \times 335 + 5$ so that after 2015 seconds the lamp is 5 seconds into a cycle at which time it is green.

Answer is (A).

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4. The panes of glass are $5x \times 2x$ for some x. Then horizontally $4 \times 2x + 5 \times 5 = 8x + 25 = s$, and vertically $2 \times 5x + 3 \times 5 = 10x + 15 = s$. Thus, $8x + 25 = 10x + 15 \Rightarrow x = 5$. Then $s = 10 \times 5 + 15 = 65$.

Answer is (B).

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5. The cat eats $\frac{1}{3} + \frac{1}{4} = \frac{7}{12}$ of a can of cat food each day. There are six cans of cat food in the box, and

$$\frac{6}{7/12} = 10 + \frac{2}{7}$$

So the box of food lasts for a bit more than 10 days. In 10 days the cat has eaten $10 \times \frac{7}{12} = 5 + \frac{5}{6}$ cans of cat food with $\frac{1}{6}$ of a can left over. On the eleventh day the new can of cat food is opened, and there is

$$1 + \frac{1}{6} - \frac{7}{12} = \frac{7}{12}$$

of a can left over, which is used on the twelfth day. If the first day is Monday, then the eighth day is Monday, the ninth day is Tuesday, the tenth day is Wednesday, the eleventh day is Thursday, and the twelfth day is Friday.

Answer is (D).

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6. Each of the given polygonal regions can be partitioned into some number m of squares each of area 1, and some number n of triangular regions each of area $\frac{1}{2}$, so a formula for the area of a typical region is $A = m + \frac{1}{2}n$. Counting m and n for each region and substituting them into the formula, we find the areas are, in order:

$$\begin{array}{llcl llcl} \text{A:} & 4(1) + 2\left(\frac{1}{2}\right) & = & 5 & \text{B:} & 4(1) + 2\left(\frac{1}{2}\right) & = & 5 \\ \text{C:} & 4(1) + 3\left(\frac{1}{2}\right) & = & 5.5 & \text{D:} & 3(1) + 3\left(\frac{1}{2}\right) & = & 4.5 \\ \text{E:} & 2(1) + 6\left(\frac{1}{2}\right) & = & 5 & & & & \end{array}$$

We conclude the largest area is that of the third listed polygon, labeled (C).

Answer is (C).

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7. Let x be the length of the rope between A and B and between C and D, and let h be the distance between the line joining points A and D and the platform, both measured in metres. If the length of the cable is 76 m then $x = \frac{1}{2}(76 - 36) = 20$ and $h = \sqrt{20^2 - 12^2} = 16$. If the length of the cable is 66 m then $x = \frac{1}{2}(66 - 36) = 15$ and $h = \sqrt{15^2 - 12^2} = 9$. Therefore, the platform will rise $16 - 9 = 7$ m.

Answer is (B).

8. Distance x, speed v, and time t are related by $x = vt \Rightarrow t = \dfrac{x}{v}$. The time which Turbo takes for his trip is $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} = \frac{23}{12}$ hours. The distance he went was $3 + 2 + 1 = 6$ kilometres, so his average speed was

$$v = \frac{x}{t} = \frac{6}{23/12} = \frac{72}{23} = 3\tfrac{3}{23}$$

Answer is (E).

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9. See Senior Preliminary, Problem 3.

Answer is (D).

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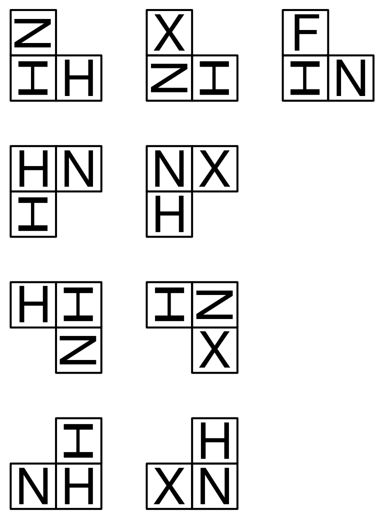
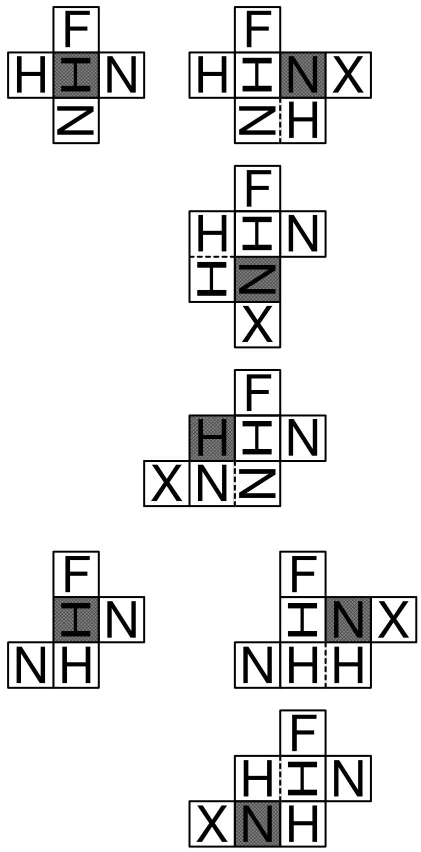
10. Only the orientation of the F is unambiguous. The H and I interchange when they are rotated through 90° as do the N and Z; the X is unchanged when it is rotated through 90°.
We can unfold the views of the cube to obtain three partial nets. There are four possible orientations for the two without an F:



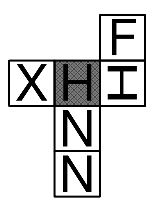
Problem 10 continued ...

. . . Problem 10 continued

Now ask how these might fit together with the partial net which contains the F. If we try to superimpose the Is, then we can eliminate any with collisions and the top net in the second column because there is a contradiction when we fold. We are left with the first column in the diagram below. Now we have to combine these with the two partial nets in the second column above. This yields the second column below.



The first and the fourth produce contradictions when we fold as does the fifth (the X lands on the F). The second and third are equivalent. We have to re-arrange it to coincide with the given net:



Answer is (A).

Junior Final, Part B

1. Observe that the units digits must be going down by 3, since $ba8$ and $6c5$ are successive terms. From the last term, $80a$, it follows that $a = 2$, and from the first term, abc, it follows that $c = 1$. Thus, $6c5$ and $80a$ are actually 615 and 802, making the common difference equal to $802 - 615 = 187$. Hence, $605 - 187 = 418$ gives $b = 4$.

   Alternative solution:

   Let d be the common difference; that is, $d = 80a - 6c5 = 6c5 - ba8 = ba8 - abc$.
   From $d = 6c5 - ba8$, we see that the ones digit of d is 7. Since $ba8 - abc = \_\_7$, we get $c = 1$.
   From $80a - 6c5 = 80a - 615 = \_\_7$, we see that $a = 2$, and therefore, we also have $d = 802 - 615 = 187$.
   We now have

$$
\begin{aligned}
187 &= 6c5 - ba8 = 615 - b28 \\
b28 &= 615 - 187 = 428,
\end{aligned}
$$

   and therefore $b = 4$.
   Thus, the required digits are $a = 2$, $b = 4$, and $c = 1$.

   Answer: $a = 2$**,** $b = 4$**, and** $c = 1$

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2. In the diagram $\ell = 139$ is the length of the longest side. Since the surface area is 2530, we have

$$
\begin{aligned}
2\left[wh + 139\,(w+h)\right] &= 2530 \\
wh + 139\,(w+h) &= 1265 \\
h\,(w + 139) &= 1265 - 139w \\
h &= \frac{1265 - 139w}{w + 139},
\end{aligned}
$$

   where both w and h are positive integers. Note that $1265 = 9 \times 139 + 14$, so that w can only take values from 1 to 9.

   By substituting positive integers for w, we find that the only possible value is $w = 2$. In this case $h = 7$, and the volume is $2 \times 7 \times 139 = 1946\,\text{cm}^3$.

   Answer: 1946 cm^3

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3. Since the 30 cm chord AB is trisected by the smaller circle, the segments AC, CD, and DB each have a length of 10 cm. Further, the segment OE bisects CD, so that CE has length 5 cm. Applying Pythagoras' theorem to triangles AEO and CEO gives

$$R^2 = 15^2 + h^2 = 225 + h^2 \;\text{ and }\; r^2 = 5^2 + h^2 = 25 + h^2$$

   Subtracting these two equations gives

$$R^2 - r^2 = 225 - 25 = 200 \Rightarrow (R - r)(R + r) = 200$$

   Since $R + r = 25$,

$$R - r = \frac{200}{25} = 8$$

   Adding $R + r = 25$ and $R - r = 8$ gives $2R = 33 \Rightarrow R = \dfrac{33}{2}$, and $r = 25 - \dfrac{33}{2} = \dfrac{17}{2}$.

   Answer: $\frac{33}{2}$ **and** $\frac{17}{2}$

4. Number the positions in the combination from 1 to 16, and give the positions of each number in the combination as an ordered pair indicating the two positions occupied by the number. Since there are not enough positions on the left, the 5 and 8 can only be placed in one way, giving:

_ _ _ 5 8 4 _ _ _ 5 _ _ _ 8 _ _

So that the 5 has positions given by $(4, 10)$ and 8 by $(5, 14)$. There are two options for the 4, which are $(1, 6)$ or $(6, 11)$. The first of these gives:

4 _ _ 5 8 4 _ _ _ 5 _ _ _ 8 _ _

Placing the 3 at $(3, 7)$ forces the following placements:

4 6 3 5 8 4 3 7 6 5 1 2 1 8 2 7

Which is one of the possible combinations. Continuing with the 4 at $(1, 6)$, neither the 1 nor the 2 can be placed in positions 2 or 3, so the 6 and 7 must be placed there. This forces the following placements:

4 6 7 5 8 4 _ _ 6 5 7 _ _ 8 _ _

With this arrangement the 3 can be placed at $(8, 12)$ or $(12, 16)$, forcing:

$$\underbrace{4\ 6\ 7\ 5\ 8\ 4\ \_\ 3\ 6\ 5\ 7\ 3\ 1\ 8\ 1\ \_}_{(8,12)} \quad \text{or} \quad \underbrace{4\ 6\ 7\ 5\ 8\ 4\ \_\ \_\ 6\ 5\ 7\ 3\ 1\ 8\ 1\ 3}_{(12,16)}$$

The 2 cannot be placed in either of these cases. Therefore, the only way the 4 can be placed at $(1, 6)$ is with the 3 at $(3, 7)$. Placing the 4 at $(6, 11)$ gives the following placements:

_ _ _ 5 8 4 _ _ _ 5 4 _ _ 8 _ _

In this arrangement the 3 can be placed at $(3, 7)$, at $(9, 13)$, or at $(8, 12)$. Placing the 3 at $(3, 7)$ gives the following placements:

_ _ 3 5 8 4 3 _ _ 5 4 _ _ 8 _ _

Then the 6 can be placed at $(2, 9)$ or $(1, 8)$, forcing:

$$\underbrace{\_\ 6\ 3\ 5\ 8\ 4\ 3\ \_\ 6\ 5\ 4\ \_\ 1\ 8\ 1\ \_}_{(2,9)} \quad \text{or} \quad \underbrace{6\ \_\ 3\ 5\ 8\ 4\ 3\ 6\ \_\ 5\ 4\ \_\ 1\ 8\ 1\ \_}_{(1,8)}$$

The 2 and the 7 cannot be placed in either case. Therefore, the 3 cannot be placed at $(3, 7)$. Placing the 3 at $(9, 13)$ gives the following placements:

_ _ _ 5 8 4 _ _ 3 5 4 _ 3 8 _ _

Then the 6 can be placed only at $(1, 8)$ which forces the following placements:

6 _ _ 5 8 4 3 6 3 5 4 2 3 8 1 2

…Problem 4 continued

In this case neither the 1 nor the 7 can be placed. Therefore, the 3 cannot be placed at $(9, 13)$, and so it must be at $(8, 12)$. This forces:

$$\underline{1}\ \underline{6}\ \underline{1}\ \underline{5}\ \underline{8}\ \underline{4}\ \underline{7}\ \underline{3}\ \underline{6}\ \underline{5}\ \underline{4}\ \underline{3}\ \underline{2}\ \underline{8}\ \underline{7}\ \underline{2}$$

This is the other possible combination.

Answer: The possible combinations are:
$$\underline{\mathbf{4}}\ \underline{\mathbf{6}}\ \underline{\mathbf{3}}\ \underline{\mathbf{5}}\ \underline{\mathbf{8}}\ \underline{\mathbf{4}}\ \underline{\mathbf{3}}\ \underline{\mathbf{7}}\ \underline{\mathbf{6}}\ \underline{\mathbf{5}}\ \underline{\mathbf{1}}\ \underline{\mathbf{2}}\ \underline{\mathbf{1}}\ \underline{\mathbf{8}}\ \underline{\mathbf{2}}\ \underline{\mathbf{7}}$$
$$\underline{\mathbf{1}}\ \underline{\mathbf{6}}\ \underline{\mathbf{1}}\ \underline{\mathbf{5}}\ \underline{\mathbf{8}}\ \underline{\mathbf{4}}\ \underline{\mathbf{7}}\ \underline{\mathbf{3}}\ \underline{\mathbf{6}}\ \underline{\mathbf{5}}\ \underline{\mathbf{4}}\ \underline{\mathbf{3}}\ \underline{\mathbf{2}}\ \underline{\mathbf{8}}\ \underline{\mathbf{7}}\ \underline{\mathbf{2}}$$

---

5. (a) We will list all possible configurations based on the number of rows.

1 row: There is exactly one way to place 9 pennies in one row; namely,

first row	9

Here, the number 9 indicates that there are 9 pennies in the first row.

2 rows: There are exactly four ways to place 9 pennies in two rows; namely,

first row	1	2	3	4
second row	8	7	6	5

Here, the first column $1, 8$ indicates that there is 1 penny in the first row and 8 pennies in the second row.

3 rows: There are exactly three ways to place 9 pennies in three rows; namely,

first row	1	1	2
second row	2	3	3
third row	6	5	4

It follows that there are eight ways to arrange 9 pennies, that is, $A(9) = 8$.

Alternative solution:

This problem deals with *partitions of integers*, that is, how to write a positive integer as the sum of smaller positive integers. Here it is required to count the number of ways to write 9 as the sum of distinct integers, without regard to order. Note that

$$9 = 9 = 8 + 1 = 7 + 2 = 6 + 3 = 6 + 2 + 1 = 5 + 4 = 5 + 3 + 1 = 4 + 3 + 2$$

Therefore, $A(9) = 8$.

Answer: See proof above.

…Problem 5 continued

(b) We will use the same notations as above and examine the smallest number k.
First note that $A(m) \leq A(n)$ for any $n > m$, since we can always add $n - m$ pennies at the last row (where the rows below have more pennies than the rows above).

- $A(1) = 1$:

first row	1

- $A(2) = 1$:

first row	2

- $A(3) = 2$:

first row	3	1
second row		2

- $A(4) = 2$:

first row	4	1
second row		3

- $A(5) = 3$:

first row	5	1	2
second row		4	3

- $A(6) = 4$:

first row	6	1	2	1
second row		5	4	2
third row				3

- $A(7) = 5$:

first row	7	1	2	3	1
second row		6	5	4	2
third row					4

- $A(8) = 6$:

first row	8	1	2	3	1	1
second row		7	6	5	2	3
third row					5	4

We just saw that $A(9) = 8$, and therefore $A(n) \neq 7$ for any $n \in \mathbb{N}$.

Answer: $k = 7$

Senior Final, Part A

1. In the regular pentagon $ABCDE$ the central angles are all equal to

$$\alpha = \frac{360}{5} = 72^\circ$$

The triangle COD is isosceles with

$$\angle CDO = \angle DCO = \beta = \frac{1}{2}(180 - \alpha) = 54^\circ$$

Then the internal angles are all equal to

$$z = 2\beta = 108^\circ$$

and the external angles are all equal to

$$y = 180 - z = 72^\circ$$

Since $FA = AB = AE$, the triangle AFE is an isosceles triangle $\angle AEF = x$ and $2x + y = 180^\circ$. Thus,

$$2x = 180^\circ - 72^\circ = 108^\circ \Rightarrow x = 54^\circ$$

Therefore, $x : y : z = 54 : 72 : 108 = 3 : 4 : 6$.

Answer is (E).

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2. See Senior Preliminary, Problem 3.

Answer is (D).

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3. A possible strategy for Al is to always leave Bev a multiple of 6 coins. He can arrange that by taking 5 coins, leaving Bev with $2010 = 6 \times 335$ coins after his first move.

Answer is (E).

---

4. If the box has dimensions $a \times b \times c$, then its surface area is $2(ab + ac + cb) = 175$ and the sum of the lengths of all its edges is $4(a + b + c) = 80 \Rightarrow a + b + c = 20$. The sum of the lengths of the four interior diagonals is $4\sqrt{a^2 + b^2 + c^2}$. Then

$$400 = (a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ac) \Rightarrow a^2 + b^2 + c^2 = 400 - 175 = 225 = 15^2$$

Therefore

$$4\sqrt{a^2 + b^2 + c^2} = 4 \times 15 = 60$$

Answer is (C).

5. Each angle in an octagon is $135^\circ = \frac{3\pi}{4}$ so the area of each circular sector is

$$\tfrac{1}{2}r^2\theta = \tfrac{1}{2}\left(3^2\right)\left(\frac{3\pi}{4}\right) = \frac{27\pi}{8}$$

and the area of all eight is 27π. Extend opposite sides of the octagon to form a square. Each of the corners is an isosceles right triangle with hypotenuse 6, so the two perpendicular legs have length $3\sqrt{2}$. Then each side of the square is

$$s = 2\left(3\sqrt{2}\right) + 6 = 6\left(1+\sqrt{2}\right)$$

and the area of the square is

$$s^2 = 36\left(1+\sqrt{2}\right)^2 = 36\left(3+2\sqrt{2}\right)$$

The area of each triangle is

$$\frac{1}{2}\left(3\sqrt{2}\right)^2 = 9$$

so the area of the octagon is

$$36\left(3+2\sqrt{2}\right) - 4\times 9 = 72\left(1+\sqrt{2}\right)$$

and the area of the shaded region is

$$72\left(1+\sqrt{2}\right) - 27\pi = 9\left(8+8\sqrt{2}-3\pi\right)$$

Answer is (A).

---

6. First note that $2015 = 9\times 223 + 8$. So the smallest integer with sum of digits 2015 will have 223 nines and an eight. The smallest will be eight followed by 223 9's. Add 1 to get 9 followed by 223 0's which has sum of digits 9.

Answer is (B).

---

7. Using the Laws of Logarithms

$$p = \log_{10}\left(2^2\times 3^3\times 4^4\times 5^5\times 6^6\right)$$

Thus

$$10^p = 2^2\times 3^3\times 4^4\times 5^5\times 6^6 = 2^{16}\times 3^9\times 5^5$$

Therefore the largest power of 2 that is a factor is 2^{16}.

Answer is (C).

8. It is always possible to produce 1, 2, 3, 4, 5, and $15 = 1 + 2 + 3 + 4 + 5$. Further, any number which can be written as a sum of four of the numbers, namely 10, 11, 12, 13, and 14 can be generated by deleting a single number. Further, since $6 + 9 = 15$ and $7 + 8 = 15$, if 6 can be generated so can 9, and if 7 can be generated so can 8. So a bad arrangement is one for which either 6 or 7 cannot be generated. Note that

$$6 = 1 + 5 = 2 + 4 = 1 + 2 + 3 \quad 7 = 2 + 5 = 3 + 4 = 1 + 2 + 4$$

The only two arrangements for which 6 cannot be generated are 12534 and 14352. But the second of these can be obtained by reflection of the first. The only two arrangements for which 7 cannot be generated are 13245 and 15423. But the second of these can be obtained by reflection of the first. From this is it clear that there are two bad arrangements.

Answer is (B).

---

9. Any shortest route from X to Y consists of eight steps, four to the right and four up. The number of routes can be counted directly, as follows:

- There is one route from X to Y through A, four up followed by four to the right.
- There are four routes from X to B (three up and one to the right).
- Similarly, there are four routes from B to Y.
- So there are $4 \times 4 = 16$ routes from X to Y through B.
- Similarly for routes through D and C.
- This gives a total of $2 \times (1 + 4 \times 4) = 34$ routes.

Alternative solution:

Call the area inside the grid where there are no black lines the "restricted area". Any route that goes through the restricted area goes through O. The total number of routes from X to Y, including those through the restricted area, equals the number of ways to choose the four steps up. This is

$$\binom{8}{4} = \frac{8!}{4!\,4!} = \frac{8 \cdot 7 \cdot 6 \cdot 5}{4 \cdot 3 \cdot 2 \cdot 1} = 70$$

The number of routes that pass through the restricted area equals the number of routes from X to O times the number of routes from O to Y. This is

$$\left[\binom{4}{2}\right]^2 = \left(\frac{4!}{2!\,2!}\right)^2 = \left(\frac{4 \cdot 3}{2 \cdot 1}\right)^2 = 6^2 = 36$$

Thus, the number of routes from X to Y that do not go through the restricted area is $70 - 36 = 34$.

Alternative solution:

The number of paths can be found by labeling each allowed vertex with a number that is the sum of the numbers on the vertices immediately below and immediately to the left, starting with 1 at X. This gives the number paths from vertex X to each point in the diagram. Therefore, there are 34 paths from X to Y.

Answer is (D).

10. Of the positive integers strictly less than 2015

- $\left\lfloor \frac{2015}{5\times 7} \right\rfloor = 57$ are divisible by 5 and 7;
- $\left\lfloor \frac{2015}{5\times 9} \right\rfloor = 44$ are divisible by 5 and 9;
- $\left\lfloor \frac{2015}{7\times 9} \right\rfloor = 31$ are divisible by 7 and 9; and
- $\left\lfloor \frac{2015}{5\times 7\times 9} \right\rfloor = 6$ are divisible by 5, 7 and 9.

Here $\lfloor x \rfloor$ is the largest integer less than or equal to x. Since the numbers that are divisible by 5, 7, and 9 are counted in each of the first three cases above, the number of positive integers strictly less than 2015 that are divisible by exactly two of 5, 7, and 9, but not all three is

$$57 + 44 + 31 - 3 \times 6 = 114$$

Answer is (B).

Senior Final, Part B

1. Let t the number of hours the helicopter flew north. Since it flew 10 km east at 50 km/hr, this leg of the trip took

$$\frac{10}{50} = \frac{1}{5}\,\text{hr}$$

Hence, it took

$$1 - \frac{1}{5} - t = \frac{4}{5} - t$$

hours to go back to where it began.

By Pythagoras' Theorem, we have

$$\begin{aligned} (80t)^2 + 10^2 &= \left[80 \times \left(\frac{4}{5} - t\right)\right]^2 \\ 80^2 \times t^2 + 10^2 &= 80^2 \times \left(\frac{4^2}{5^2} - 2 \times \frac{4}{5} \times t + t^2\right) \end{aligned}$$

Dividing through by 10^2 and solving for t we get

$$\begin{aligned} 8^2 \times t^2 + 1 &= \frac{8^2 \times 4^2}{5^2} - 8^2 \times 2 \times \frac{4}{5} \times t + 8^2 \times t^2 \\ \Rightarrow 1 &= \frac{8^3 \times 2}{5^2} - \frac{8^3}{5} \times t \\ \Rightarrow \frac{2^9}{5} \times t &= \frac{2^{10} - 5^2}{5^2} \\ \Rightarrow t &= \frac{1024 - 25}{5} \times \frac{1}{2^9} = \frac{999}{2560} \end{aligned}$$

It follows that the helicopter flew $\frac{999}{2560} \times 80 = \frac{999}{32}$ km to the north.

Alternative solution:

Let d be the distance in kilometres that the helicopter flew north. Using Pythagoras' theorem on the return leg of the trip and the fact that the total trip took one hour gives:

$$\frac{d}{80} + \frac{10}{50} + \frac{\sqrt{d^2 + 10^2}}{80} = 1$$

Multiplying by 80 and isolating the square root gives:

$$\sqrt{d^2 + 100} = 64 - d$$

Now square both sides to obtain

$$d^2 + 100 = 64^2 - 128d + d^2 \Rightarrow 128d = 64^2 - 100 = 4\left(2^{10} - 25\right) = 4 \times 999$$

Solving for d gives

$$d = \frac{999}{32}\,\text{km}$$

Answer: $\frac{999}{32}$ **km**

2. See Junior Final Part B, Problem 4.

Answer: The possible combinations are: $\frac{4\ 6}{1\ 6}$ $\frac{3\ 5}{1\ 5}$ $\frac{8\ 4}{8\ 4}$ $\frac{3\ 7}{7\ 3}$ $\frac{6\ 5}{6\ 5}$ $\frac{1\ 2}{4\ 3}$ $\frac{1\ 8}{2\ 8}$ $\frac{2\ 7}{7\ 2}$

3. The number of $i \times j$ rectangles that can be formed using four of the nodes in the 4×4 lattice is $(5-i)(5-j)$. The table below shows the number of rectangles of each size that can be formed:

Size	Number
1×1	$4 \times 4 = 16$
1×2 & 2×1	$2 \times 4 \times 3 = 24$
1×3 & 3×1	$2 \times 4 \times 2 = 16$
1×4 & 4×1	$2 \times 4 \times 1 = 8$
2×2	$3 \times 3 = 9$
2×3 & 3×2	$2 \times 3 \times 2 = 12$
2×4 & 4×2	$2 \times 3 \times 1 = 6$
3×3	$2 \times 2 = 4$
3×4 & 4×3	$2 \times 2 \times 1 = 4$
4×4	$1 \times 1 = 1$
Total	100

Therefore, 100 rectangles can be formed on the 4×4 lattice. **Alternative solution:**
Consider the 4×4 lattice above. A rectangle is completely determined by choosing two points from each of the sides. This is equivalent to choosing two pairs of integers from 0 to 4, inclusive. There are $C(5,2)$ ways of choosing each pair of integers. By the multiplication principle, there are then

$$\left(\binom{5}{2}\right)^2 = \left(\frac{5 \times 4}{2}\right)^2 = \frac{5^2 \times 4^2}{4} = 100$$

rectangles that can be formed on the 4×4 lattice.

Answer: 100 rectangles

4. Label the vertices of the triangle A, B, and C. (See the diagram.) First note that the sum of the areas of the two smaller semicircles on sides AC and BC of the triangle is:

$$\frac{\pi}{2}\left(\frac{a}{2}\right)^2 + \frac{\pi}{2}\left(\frac{b}{2}\right)^2 = \frac{\pi}{2}\left(\frac{a^2+b^2}{4}\right)$$

But ABC is a right triangle with right angle at C, so that

$$a^2 + b^2 = (2r)^2 = 4r^2$$

Hence, the sum of the areas of the two smaller semicircles is

$$\frac{\pi}{2}r^2 = \text{area of the large semicircle, on side } AB \text{ of the triangle}$$

Now let A_a and A_b be the areas of the two lunar regions, over the sides of length a and b, respectively. Let A_{AC} and A_{BC} be the areas of the two circular sectors on sides AC and BC, respectively. Finally, let A_{ABC} be the area of the triangle ABC. Since the sum of the areas of the two smaller semicircles equals the area of the large semicircle, we have

$$(A_a + A_{BC}) + (A_b + A_{AC}) = A_{BC} + A_{AC} + A_{ABC}$$

Canceling A_{BC} and A_{AC} gives

$$A_a + A_b = A_{ABC}$$

Which is what was to be proved.

Answer: See proof above.

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5. We find restrictions on the values of m so a line through $Q = (20, 14)$ does intersect the parabola. A line through Q with slope m intersects the parabola P at the point (c, c^2) if and only if $m = \dfrac{c^2 - 14}{c - 20}$. Rearranging, we see this is equivalent to the quadratic equation $c^2 - mc + (20m - 14) = 0$, which has real solutions if and only if

$$(-m)^2 - 4(1)(20m - 14) = m^2 - 80m + 56 \geq 0$$

Solving this last inequality for m using the quadratic formula, we find $m \leq 40 - 2\sqrt{386} = r$ or $s = 40 + 2\sqrt{386} \leq m$. These are the conditions under which the line **does** intersect the parabola. So the line **does not** intersect the parabola if

$$40 - 2\sqrt{386} = r < m < s = 40 + 2\sqrt{386}$$

The sum of the values of r and s is 80. Note that the solutions to the quadratic are not required, since the sum of the roots of the quadratic $x^2 + bx + c$ is simply $-b = -(-80) = 80$.

Alternative solution:

The slopes r and s can be found by determining the points (x, x^2) on the parabola where the line from the point $(20, 14)$ is tangent to the parabola. Using calculus the slope of the tangent line to the parabola at the point (x, x^2) is $2x$. So it is only necessary to solve the equation

$$2x = \frac{x^2 - 14}{x - 20} \Rightarrow x^2 - 40x + 14 = 0$$

Which is the same as the equation above with $m = 2x$.

Answer: $r + s = 80$