BRITISH COLUMBIA COLLEGES
HIGH SCHOOL MATHEMATICS CONTEST
2001 SOLUTIONS

Junior Preliminary

1. Let r_s be the average speed on his trip to the seashore. Then

$$r_s = \frac{150 \text{ km}}{3\frac{1}{3} \text{ hr}} = 45 \text{ km/hr}$$

On the other hand, his average speed for the entire trip is

$$r = \frac{150 \text{ km} + 150 \text{ km}}{3\frac{1}{3} \text{ hr} + 4\frac{1}{6} \text{ hr}} = 40 \text{ km/hr}$$

Clearly, r_s exceeds r by 5 km/hr. <u>Answer is (**a**)</u>

2. Let the value of a pup be p, of a pooch be c, of a mutt be m, and of a bird dog be b. Then we are given:

$$p = c + m \tag{1}$$
$$b = p + c \tag{2}$$
$$2b = 3m \tag{3}$$

We must express p in terms of c. Equation (1) above can be arranged as $m = p - c$. Substituting this and (2) into (3), we have

$$\begin{aligned} 2(p + c) &= 3(p - c) \\ 2p + 2c &= 3p - 3c \\ 5c &= p \end{aligned}$$

Thus a pup is worth 5 pooches.

Alternate approach:

3 pups are worth 3 pooches + 3 mutts   from (1)
3 pups are worth 3 pooches + 2 birddogs   from (3)
3 pups are worth 3 pooches + (2 pups + 2 pooches)   from (2)

Therefore, 1 pup is worth 5 pooches. <u>Answer is (**c**)</u>

3. Since all of the possible solutions are integers, we know that x itself must be a perfect square. Then

$$\begin{aligned} x = 9 &\Longrightarrow x + \sqrt{x} = 12 \qquad \text{too small} \\ x = 16 &\Longrightarrow x + \sqrt{x} = 20 \\ x = 25 &\Longrightarrow x + \sqrt{x} = 30 \\ x = 36 &\Longrightarrow x + \sqrt{x} = 42 \\ x = 49 &\Longrightarrow x + \sqrt{x} = 56 \\ x = 64 &\Longrightarrow x + \sqrt{x} = 72 \\ x = 81 &\Longrightarrow x + \sqrt{x} = 90 \\ x = 100 &\Longrightarrow x + \sqrt{x} = 110 \\ x = 121 &\Longrightarrow x + \sqrt{x} = 132 \qquad \text{too large} \end{aligned}$$

Since the above cover cases all the possibilities in the range of values we are given, we conclude that 60 cannot be written in the required form.

Alternate approach: First recognize that $x + \sqrt{x} = \sqrt{x}(\sqrt{x} + 1)$, which is the product of two consecutive integers. Since $20 = 4 \times 5$, $30 = 5 \times 6$, $90 = 9 \times 10$, and $110 = 10 \times 11$, we see that 60 is the answer, since it canNOT be written be as the product of two consecutive integers. Answer is (**c**)

4. Because $DC = 8$ we have $AE = BF = 4$, since $ABCD$ is a square and E and F are midpoints of the sides AD and BC, respectively. The lower shaded region has area $\frac{1}{2}\pi(4)^2 = 8\pi$. The upper shaded region is comprised of a 4×8 rectangle with two quarter circles removed; thus it has area $32 - 2\left[\frac{1}{4}\pi(4)^2\right] = 32 - 8\pi$. When we sum these two values we get a total area of 32 square units.

   Alternate approach: One can partition the region in two with a vertical line through the middle of the diagram. The in each of the regions we easily see that the two shaded pieces can be put together to make a square of side 4. Since we would then have two such squares, the total area of the shaded pieces is $16 + 16 = 32$. Answer is (**b**)

5. $c^{4d} - 5 = (c^d)^4 - 5 = 3^4 - 5 = 81 - 5 = 76$. Answer is (**d**)

6. The single amoeba becomes two after 3 minutes. Thus 57 more minutes are required for the two amoeba to fill the dish. Answer is (**e**)

7. Since the sum of its digits is a multiple of 3, so 2001 is a multiple of 3, i.e. $2001 = 3 \times 667$. The question remains whether 667 is prime or not. This requires successive testing by primes 7, 11, 13 17, 19 and 23, with success only for 23. Thus $2001 = 3 \times 23 \times 29$, and $3 + 23 + 29 = 55$. Answer is (**a**)

8. Since AD is a diameter, we see that $\angle AED = \angle ACD = 90°$. Since $AE = ED$, this further implies that $\angle EAD = \angle EDA = 45°$. Let O be the centre of the circle (which means O is the midpoint of AD). Then $\angle AOB = \angle BOC = \angle COD$ since equal length chords subtend equal angles at the centre of a circle, whence each of these three angles must be 60°. Since an isosceles triangle with a vertex of 60° must also have base angles of 60°, we conclude that $\triangle COD$ is equilateral. Since the angle subtended by a chord on the circumference of a circle is half of the angle subtended at the centre of the circle, we deduce that $\angle CAD = 30°$ (one could also draw this conclusion by noting that $\triangle ACD$ is 30°-60°-90° triangle). Thus $\angle CAE = 30° + 45° = 75°$. Answer is (**c**)

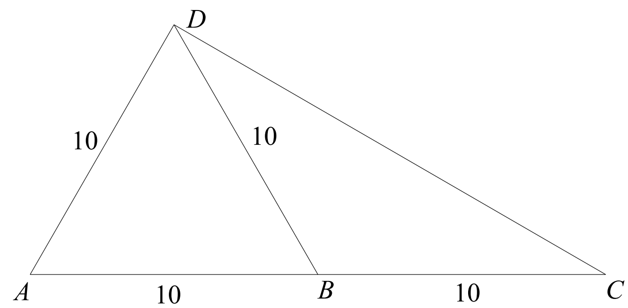
9. To be divisible by 11 the sum of the digits in the odd positions less the sum of the digits in the even positions must be a multiple of 11. Letting a be the missing (leading) digit of the number, we see that the difference in these two sums is $(a + 6 + 3) - (5 + 7 + 4) = a - 7$. In order for this to be a multiple of 11 and a to be a single decimal digit, we require $a = 7$.

   Alternate approach: We can work backwards in the multiplication of some (5-digit) number by 11 which produces a 6-digit number whose last 5 digits are 56734. It is easily seen that the 5-digit number must 68794, which on multiplication by 11 yields 756734. Answer is (**d**)

10. Clearly the points A, B, C, and D must be aligned as in the diagram below. Thus $\triangle ABD$ is equilateral and $\triangle BCD$ is isosceles with vertex angle 120° and base angles 30°. This implies that $\angle ADC = 60° + 30° = 90°$. By applying the Theorem of Pythagoras to $\triangle ADC$ we see that

$$DC = \sqrt{20^2 - 10^2} = \sqrt{300} = 10\sqrt{3}$$

Alternate approach: From the diagram we see that a circle of radius 10 centred at B passes through A, C, and D. Thus $\angle ADC = 90°$. Then we can use the Theorem of Pythagoras as above to finish the solution. Answer is (**c**)



11. Let the first digit of D be a and the second digit of D be b. Then $D = 10a + b$. We are told that

$$\begin{aligned} \frac{D}{2} - \frac{D}{3} &= a + b \\ \frac{D}{6} &= a + b \\ 10a + b &= 6a + 6b \\ 4a &= 5b \end{aligned}$$

Since both a and b are single decimal digits, we conclude that either $(a, b) = (0, 0)$ or $(a, b) = (5, 4)$. The first choice does not produce a two digit number D. Thus we must have the second choice, which means that $a + b = 9$. Answer is (**c**)

12. Let x be the distance from P to A and let y be the distance from A to C. Then the distance traveled by each hiker is: $x + y = 3 + 4 = 7$ km. Thus $y = 7 - x$. On the other hand, the Theorem of Pythagoras yields

$$\begin{aligned} (7 - x)^2 &= 4^2 + (x + 3)^2 = 16 + x^2 + 6x + 9 = x^2 + 6x + 25 \\ 49 - 14x + x^2 &= x^2 + 6x + 25 \\ 24 &= 20x \\ x &= 1\tfrac{1}{5} \end{aligned}$$

Answer is (**d**)

13. Let the sides of the triangle have lengths a, b, and c, where $a + b + c = 10$. We may certainly assume that $a \leq b \leq c$. We also need to notice that for three lengths to form a triangle, the longest side must be shorter than the sum of the two shorter sides, i.e. $c < a + b$. There are two approaches one use at this point.

First approach: The information above implies that $c < 5$, or $c \leq 4$. On the other hand, $10 = a + b + c \leq 3c$, or $c > 3$, i.e. $c \geq 4$. Therefore, we must have $c = 4$ and $a + b = 6$ with $a \leq b \leq 4$. There are two possibilities for a and b, namely $a = b = 3$ and $a = 2$, $b = 4$.

Second approach: Let us instead simply list all the sets of integers satisfying $a + b + c = 10$ and $a \leq b \leq c$. For each set we will examine the relationship $c < a + b$. If it is false, we do not have a triangle; if it is true, we do have a triangle.

a	b	c	$c < a + b$
1	1	8	False
1	2	7	False
1	3	6	False
1	4	5	False
2	2	6	False
2	3	5	False
2	4	4	TRUE
3	3	4	TRUE

Thus there are only two such triangles.

14. Let the number of students in rooms A, B, and C be a, b, and c, respectively. Then we have $a + b + c = 100$. In addition we get

$$\begin{aligned} a - \tfrac{1}{2}a \qquad\quad + \tfrac{1}{3}c &= a \\ b + \tfrac{1}{2}a - \tfrac{1}{5}b \qquad\quad &= b \\ c \qquad\quad + \tfrac{1}{5}b - \tfrac{1}{3}c &= c \end{aligned}$$

These lead to $c = \frac{3}{2}a$ and $b = \frac{5}{2}a$. Thus

$$\begin{aligned} a + \frac{3}{2}a + \frac{5}{2}a &= 100 \\ 5a &= 100 \\ a &= 20 \end{aligned}$$

Alternate approach: Since the number of students in each room stays the same after each period, the number of students who move from one room to another must all be the same. That is,

$$\tfrac{1}{2}a = \tfrac{1}{5}b = \tfrac{1}{3}c$$

which means that $a : b : c = 2 : 5 : 3$. To maintain these proportions in a student body of 100, we need $(a, b, c) = (20, 50, 30)$ Answer is (**b**)

15. When $n = 2$, the product is $\frac{3}{4}$, which eliminates (a). When $n = 3$ the product is $\frac{3}{4}\cdot\frac{8}{9} = \frac{2}{3}$ which eliminates (b) and (e). When $n = 4$ the product is $\frac{3}{4}\cdot\frac{8}{9}\cdot\frac{15}{16} = \frac{5}{8}$ which eliminates (c). A more sophisticated approach can also be taken: Factor each term using difference of squares:

$$\begin{aligned} P_n &= \left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{9}\right)\cdots\left(1 - \frac{1}{n^2}\right) \\ &= \left(1 - \frac{1}{2}\right)\left(1 + \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 + \frac{1}{3}\right)\cdots\left(1 - \frac{1}{n}\right)\left(1 + \frac{1}{n}\right) \\ &= \left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{2}{3}\right)\left(\frac{4}{4}\right)\cdots\left(\frac{n-1}{n}\right)\left(\frac{n+1}{n}\right) \\ &= \frac{1}{2}\cdot\frac{n+1}{n} = \frac{n+1}{2n} \end{aligned}$$

Answer is (**d**)

Senior Preliminary

1. This problem is the same as #6 on the Junior Preliminary. Answer is (**d**)

2. Converting both 16 and 8 to powers of 2, the equation becomes:

$$\begin{aligned}(2^4)^{2x+\frac{1}{4}} &= (2^3)^{3+2x} \\ 2^{8x+1} &= 2^{9+6x} \\ 8x+1 &= 9+6x \\ 2x &= 8 \\ x &= 4\end{aligned}$$

Answer is (**b**)

3. When we substitute $(0,5)$ for (x,y) we get $c=5$. Substituting the other points then yields:

$$\begin{aligned}12 &= a-b+5 \\ -3 &= 4a+2b+5\end{aligned} \qquad \text{or} \qquad \begin{aligned}a-b &= 7 \\ 4a+2b &= -8\end{aligned}$$

Taking twice the first of these and adding to the second we get $6a=6$, or $a=1$, whence $b=-6$. Thus $a+b+c=1-6+5=0$. Answer is (**c**)

4. The centre of the circle is located at the midpoint between $(-5,12)$ and $(5,12)$, namely $(0,12)$. So the equation of the circle is

$$x^2+(y-12)^2=5^2=25$$

Of the 5 possibilities, only $(0,7)$ satisfies this equation.

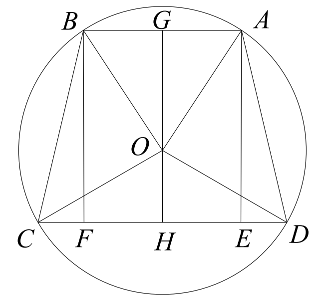
Alternate approach: If one draws the circle (even a rough sketch will work), and plots the 5 distractor points, all but $(0,7)$ are nowhere near the circle. It is then a simple to verify that $(0,7)$ lies on the circle by using the distance formula. Answer is (**b**)

5. Let n be the number of students in the class. The number of classmates for each of Chris and Pat is then $n-1$. Let us now compare the fractions $\frac{12}{17}$ and $\frac{5}{7}$. We see that they are respectively equal to $\frac{84}{119}$ and $\frac{85}{119}$. Since $\frac{12}{17}<\frac{5}{7}$, we see that Pat must be a woman and Chris a man. If we let w be the number of women in the class, we now have two relations:

$$\begin{aligned}\frac{w-1}{n-1} &= \frac{12}{17} \\ 17w-17 &= 12n-12 \\ 17w-12n &= 5\end{aligned} \qquad \text{and} \qquad \begin{aligned}\frac{w}{n-1} &= \frac{5}{7} \\ 7w &= 5n-5 \\ 7w-5n &= -5\end{aligned}$$

Multiplying the left equation by 7 and the right by 17, and subtracting we get $n=120$.

Alternate approach: Since each fraction applied to the set of classmates of either Chris or Pat must yield an integer, we know that the number of classmates must be a multiple of the least common multiple of 7 and 17, namely 119. The number of students in the class must be 1 greater than this multiple. The only possibility among the candidates is 120.
Answer is (**d**)



6. Drop perpendiculars from A and B to CD meeting CD at E and F, respectively. Then $EF = AB = 2$. Thus $DE = CF = \frac{1}{2}\cdot 6 = 3$. Using the Theorem of Pythagoras on $\triangle ADE$ and $\triangle BCF$ we see that the altitude of the trapezoid is $AE = BF = 4$. Now run a vertical line through the centre, O, of the circle, meeting AB at G and CD at H, and join O to each of A, B, C, and D. Then $AG = BG = 1$ and $CH = DH = 4$. Setting r equal to the radius of the circle and using the Theorem of Pythagoras on $\triangle GOB$ and $\triangle HOC$, we see that

$$\begin{aligned} GO &= \sqrt{r^2 - 1} \\ HO &= \sqrt{r^2 - 16} \end{aligned}$$

Since $GO \pm HO = 4$ (the plus/minus possibility occurs because we do not know if O lies inside or outside the trapezoid), we have

$$\begin{aligned} \sqrt{r^2-1} \pm \sqrt{r^2-16} &= 4 \\ \sqrt{r^2-1} - 4 &= \mp\sqrt{r^2-16} \\ r^2 - 1 - 8\sqrt{r^2-1} + 16 &= r^2 - 16 \\ 31 &= 8\sqrt{r^2-1} \\ 961 &= 64(r^2-1) \\ 1025 &= 64r^2 \end{aligned}$$

from which we see that $r = \frac{1}{8}\sqrt{1025}$.

Alternate approach: Borrowing the notation, diagram, and some of the development above, and letting P denote the centre of the circle, we set $x = PF$. Thus $PG = |4-x|$ and $PA = r$. From applying the Theorem of Pythagoras to $\triangle PFA$ and $\triangle PGC$, respectively, we obtain:

$$\begin{aligned} x^2 + 1^2 &= r^2 \qquad (1) \\ (4-x)^2 + 4^2 &= r^2 \end{aligned}$$

Subtracting the first from the second we get:

$$\begin{aligned} \left[(4-x)^2 - x^2\right] + 4^2 - 1 &= 0 \\ (16 - 8x) + 15 &= 0 \\ 31 &= 8x \end{aligned}$$

from which we get $x = 31/8$. Using this value in (1) we get

$$r^2 = \left(\frac{31}{8}\right)^2 + 1 = \frac{31^2 - 8^2}{8^2} = \frac{1025}{8^2}$$

from which we get $r = \frac{1}{8}\sqrt{1025}$. Answer is **(e)**

7. First we observe that $(x+y)^2 = x^2 + y^2 + 2xy$. Thus $1^2 = 4 + 2xy$, or $xy = -\frac{3}{2}$. Now

$$\begin{aligned} x^3 + y^3 &= (x+y)^3 - 3x^2y - 3xy^2 \\ &= 1^3 - 3xy(x+y) \\ &= 1 - 3\cdot\left(-\tfrac{3}{2}\right)\cdot 1 \\ &= 1 + \tfrac{9}{2} = \tfrac{11}{2} \end{aligned}$$

Alternate approach: Using the above to find $xy = -\frac{3}{2}$, one could then express $x^3 + y^3$ by factoring:

$$x^3 + y^3 = (x+y)(x^2 - xy + y^2) = 1(4 - xy) = 4 + \tfrac{3}{2} = \tfrac{11}{2}$$

Answer is (**c**)

8. Let the x-intercepts of the two lines be $(a, 0)$ and $(2a, 0)$. The lines have slopes of $2/(9-a)$ and $2/(9-2a)$, respectively. Since we are dealing with perpendicular lines, one of these is the negative reciprocal of the other, i.e.

$$\begin{aligned} \frac{2}{9-a} &= -\frac{9-2a}{2} \\ 4 &= -(81 - 27a + 2a^2) \\ 2a^2 - 27a + 85 &= 0 \\ (2a-17)(a-5) &= 0 \end{aligned}$$

which means that $a = 5$ or $a = 17/2$. The sum of the x-intercepts is $3a$, which is either 15 or $51/2$. Only the latter is found among the list of possibilities. Answer is (**c**)

9. AB is the diameter of the larger circle since the centre C lies on AB. Since the two circles are tangent to each other at A, we see that AD is thus a diameter of the smaller circle. Now $\angle AED = 90°$ since AD is a diameter of the smaller circle. Since FC is perpendicular to AD we see that triangles ECD and ACE are similar. Then

$$\frac{AC}{EC} = \frac{EC}{CD}$$

Setting R to be the radius of the large circle we get

$$\begin{aligned} \frac{R}{R-5} &= \frac{R-5}{R-9} \\ R^2 - 9R &= R^2 - 10R + 25 \\ R &= 25 \end{aligned}$$

Then the diameter of the smaller circle is $2R - 9 = 41$, whence its radius is $20\frac{1}{2}$.

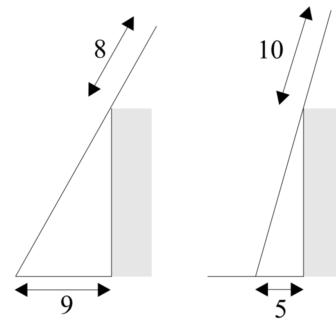
Answer is (**d**)

10. Each of the 9 granddaughters of the woman has exactly 6 cousins. Thus it appears that we have a total of $9 \times 6 = 54$ pairs of cousins. However, each distinct pair of cousins has been counted twice in this enumeration (once from the perspective of each cousin). Thus there are only half that many distinct pairs of cousins.

Alternate approach: Among the 9 granddaughters there are $\frac{9 \times 8}{2} = 36$ pairs of girls, but in the three triples of sisters there are 9 pairs of sisters (3 such pairs among each set of sisters), which need to be excluded. Thus there are $36 - 9 = 27$ pairs of cousins. Answer is (**c**)

11. When we plot the quadrilateral we see that the figure is a parallelogram with base length k (along the x-axis) and altitude n. Thus the area is kn. Answer is (**d**)

12. Let L be the length of the ladder in metres and let h be the height of the wall in metres. In the first scenario we have a right-angled triangle with base 9 m, height h m, and hypotenuse $L-8$ m. In the second scenario we have base 5 m, height h m, and hypotenuse $L-10$ m. Thus



$$\begin{aligned}(L-8)^2-9^2 &= h^2 = (L-10)^2-5^2\\ L^2-16L+64-81 &= L^2-20L+100-25\\ 4L &= 92\\ L &= 23\end{aligned}$$

Then $h^2=(23-10)^2-5^2=13^2-25=144$, implying that $h=12$ m. Answer is (**b**)

13. To obtain the area of the shaded region we will compute the sum of the areas of triangle ABC, the area of the semicircle on AB, and the area of the semicircle on AC, and then we will subtract from this sum the area of the semicircle on BC. Let a, b, and c be the lengths of the sides BC, AC, and AB, respectively. Note that $a=10$. Then our desired area $\mathcal{A}$ is:

$$\mathcal{A}=\tfrac{1}{2}\cdot 10\cdot 4+\tfrac{1}{2}\pi c^2+\tfrac{1}{2}\pi b^2-\tfrac{1}{2}\pi 10^2=20+\tfrac{1}{2}\pi(b^2+c^2)-50\pi=20+\tfrac{1}{2}\pi a^2-50\pi=20$$

Alternate approach: Let the areas of the three semicircles be A, B, and C with C being the largest. Let T be the area of the triangle. Then the area of the shaded region is $a+b-[c-t]=(a+b-c)+t$. By the Theorem of Pythagoras we know $c=a+b$. Therefore, the area of the shaded region is $t=\frac{1}{2}\cdot 10\cdot 4=20$. Answer is (**e**)

14. Note

$$\begin{aligned}f_0(x) &= \frac{1}{1-x}\\ f_1(x) &= f_0\left(\frac{1}{1-x}\right)=\frac{1}{1-\dfrac{1}{1-x}}=\frac{x-1}{x}\\ f_2(x) &= f_0\left(\frac{x-1}{x}\right)=\frac{1}{1-\dfrac{x-1}{x}}=x\end{aligned}$$

Thus $f_3(x)=f_0(x)$, and the functions begin to cycle, i.e. $f_{n+3k}=f_n$. Therefore, we see that

$$f_{100}(3)=f_{1+99}(3)=f_1(3)=\frac{3-1}{3}=\frac{2}{3}$$

Alternate approach: The above approach looks at the behaviour of the function for all values x. We need only trace what happens when we start with $x=3$. Thus $f_0(3)=-\frac{1}{2}$, $f_1(3)=\frac{2}{3}$, $f_2(3)=3$, and $f_3(3)=f_0(3)$. At this point we begin to repeat. Thus $3=f_2(3)=f_5(3)=\cdots=f_{98}(3)$. Similarly, $f_{99}(3)=f_0(3)$ and $f_{100}(3)=f_1(3)=\frac{2}{3}$. Answer is (**b**)

15. It is easy to verify that both 1 and 2 are true. It remains to check 3. Suppose first that $a\le b$ and $a\le c$. Then both sides yield a, and statement 3 is true. Next suppose that $a\ge b$ and $a\ge c$. Then both sides will yield the larger of b or c, and statement 3 is true. If a lies strictly between b and c, then the left hand side clearly yields a; the right hand side then yields the larger of a and whichever of b or c is smaller (which is also smaller than a. Thus both sides yield a and statement 3 is again true. Thus statement 3 is true in all cases. So all three statements are true. Answer is (**e**)

Junior Final - Part A

1. If $n = 3k \pm 1$, then $n^2 + 2 = 9k^2 \pm 6k + 3 = 3(3k^2 \pm 2k + 1)$, which is not prime. Thus we must have n a multiple of 3 in order to have $n^2 + 2$ prime. This eliminates all but 81. If we check 81, we see that 79 and 83 are both prime.
   This problem can also be solved simply by testing each possibility to see if both $n^2 - 2$ and $n^2 + 2$ are prime. Answer is (**c**)

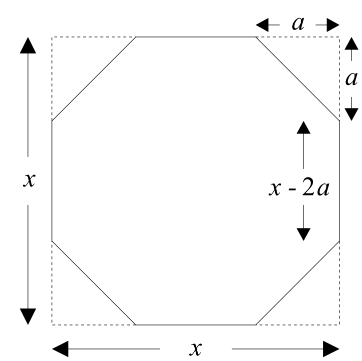
2. Let the radius of c be r. Then the radii of b and a are $2r$ and $4r$, respectively. Therefore, the areas of a, b, and c are $16\pi r^2$, $4\pi r^2$, and πr^2, respectively. Then the area of the shaded region is $4\pi r^2 - \pi r^2 = 3\pi r^2$, and the area of the unshaded region is $16\pi r^2 - 3\pi r^2 = 13\pi r^2$. The ratio of shaded to unshaded areas is then $3 : 13$. Answer is (**a**)

3. Clearly the number of cars entering the diagram must equal the number of cars exiting the diagram, i.e.

$$\begin{aligned} 200 + 180 + 70 + 200 &= 20 + 30 + W + 400 \\ 650 &= W + 450 \\ W &= 200 \end{aligned}$$

Answer is (**b**)

4. Let the length of the removed corner piece be a (see diagram at the right). Then a side of the resulting octagon is equal to $x - 2a$. Using the Theorem of Pythagoras on the right-angled triangle in any corner gives us:



$$\begin{aligned} (x - 2a)^2 &= a^2 + a^2 = 2a^2 \\ x - 2a &= a\sqrt{2} \\ x &= a(2 + \sqrt{2}) \\ a &= \frac{x}{2 + \sqrt{2}} \end{aligned}$$

We are interested in the length of the side of the octagon:

$$\begin{aligned} x - 2a &= x - \frac{2x}{2 + \sqrt{2}} = \frac{2x + x\sqrt{2} - 2x}{2 + \sqrt{2}} = \frac{x\sqrt{2}}{2 + \sqrt{2}} \\ &= \frac{x\sqrt{2}(2 - \sqrt{2})}{4 - 2} = \frac{x(2\sqrt{2} - 2)}{2} = x(\sqrt{2} - 1) \end{aligned}$$

Alternate approach: Let b be the side length of the regular octagon. Since the removed corners are 45°-45°-90° triangles, the legs have length $b/\sqrt{2}$. Thus

$$\begin{aligned} x &= \frac{2b}{\sqrt{2}} + b = \frac{2 + \sqrt{2}}{\sqrt{2}} b \\ \text{or} \quad b &= \frac{\sqrt{2}}{2 + \sqrt{2}} x \end{aligned}$$

Rationalizing the denominator we get:

$$b = \frac{\sqrt{2}}{2 + \sqrt{2}} \cdot \frac{2 - \sqrt{2}}{2 - \sqrt{2}} x = \left(\frac{2\sqrt{2} - 2}{4 - 2} \right) = (\sqrt{2} - 1)x$$

Answer is (**d**)

5. Let $x = 0.0\overline{1}$. Then we note that $10x = 0.\overline{1}$, which can also be written as $10x = 0.1\overline{1}$. Comparing this form of $10x$ with x we see that the decimal fraction expansions agree except for the first digit following the decimal point. Thus we may subtract to obtain $9x = 0.1$, which means that $x = 1/90$. Then

$$(0.0\overline{1})^{-1} + 1 = \left(\frac{1}{90}\right)^{-1} + 1 = 90 + 1 = 91.$$

Alternate method: We first note that $0.01 < x < 0.02$ or $\frac{1}{100} < x < \frac{1}{50}$, which means that $100 > \frac{1}{x} > 50$, and there is only one possible answer in this range. Answer is (**e**)

6. In order to cover the addresses from 1 to 99, we need 20 of each non-zero digit and 9 zeroes. From 100 to 199, we will have the greatest call on the digit 1 since every such address will have at least one digit 1 in it. So let us examine only the digit 1 first. From 100 to 109 we use 11 ones; from 110 to 119 we use 21 ones; for each subsequent group of ten (up to the address 199) we use a further 11 ones. Thus we want $20 + 11 + 21 + k(11) \leq 100$, implying that $k \leq 4$. That is, up to address 159 we have $20+11+21+4(11) = 96$ ones. Address 160, 161, and 162 use up a further 4 ones and we have exhausted the 100 ones we started with. Thus the first address which cannot be displayed is 163. Answer is (**d**)

7. If we first consider $p = 2$ and $q = 4$, we easily see that we have $1 + 2 + 3 = 6$ points of intersection. If we now consider $p = 3$ and $q = 4$ we see that by considering any pair of the $p = 3$ dots, together with the $q = 4$ dots opposite, we get 6 points of intersection. Now there are three such distinct pairs which gives us a total of 18 points of intersection.
It is expected that most students will simply draw a picture and enumerate all the points of intersection. Answer is (**c**)

8. Let the first-mentioned population be n. Then $n = a^2$ for some integer a. We then also have $n + 100 = b^2 + 1$ and $n + 200 = c^2$ for some integers b and c. That is, $a^2 + 99 = b^2$ and $a^2 + 200 = c^2$, or $b^2 - a^2 = 99$ and $c^2 - a^2 = 200$. Adding these together we get $c^2 - b^2 = 101$. Thus $(c - b)(c + b) = 101$. Since 101 is prime we see that $c - b = 1$ and $c + b = 101$, whence $c = 51$ and $b = 50$. Thus $n = a^2 = b^2 - 99 = 2401 = 49^2 = 7^4$. Answer is (**b**)

9. Let a be the number of adults, t be the number of teenagers, and c be the number of children attending the movie. Then $a + t + c = 100$ is the number of persons attending the movie, and $3a + 2t + c/4 = 100$ is the number of dollars taken in by the movie house. Multiplying the second equation by 4 to clear the fractions, and subtracting the first equation we get: $11a + 7t = 300$, or $t = (300 - 11a)/7$. Since we seeking integer solutions and we want the smallest possible value for a, we may simply examine successive values of a starting with $a = 0$ until we find an integer solution for t. The first (i.e. smallest) value of a is $a = 5$, which gives $t = 35$ (and $c = 60$). Answer is (**c**)

10. We can first make a shaded "chain" to the coast by shading one coastal region at the top left and one of the two interior unshaded regions linking the two shaded regions. This requires 2 conversions. This shaded chain of 7 states can now be unshaded one at a time working from the interior to the coast, requiring another 7 conversions for a total of 9 conversions. To see that there can never be fewer than 9 conversions, we note first that we must convert the shaded states to unshaded in order to minimize the number of conversions, and secondly that it is necessary to convert at least one unshaded coastal state and one unshaded interior

state to shaded in order to avoid a shaded state being ultimately surrounded by unshaded states. This means that we would have a minimum of 7 shaded states to be converted to unshaded (in addition to the minimum of 2 unshaded that need to be converted to shaded). Thus we require at least 9 conversions. So 9 is the minimum number of conversions needed to pacify Aresia. Answer is (**c**)

Junior Final - Part B

1. Let n be the 3-digit integer in question. Clearly n is a multiple of 7. Since it has 3 digits, we may start with the smallest 3-digit multiple of 7 and examine successive multiples of 7 until the conditions are satisfied. The first 3-digit multiple of 7 is 105, which is also a multiple of 5; the next is 112, which is a multiple of 2; the next is $119 = 7 \times 17$, and this leaves a non-zero remainder when divided by any of 2, 3, 4, 5, or 6. Thus $n = 119$.

   Alternate Solution: There was at least one student who, in reading the problem, recognized (unlike the problem posers!) that nowhere is there a mention that the smallest 3-digit integer had to be positive. Since any negative number is smaller than any positive one, the student then found the smallest negative 3-digit integer satisfying the conditions. Since 994 is a multiple of 7, so is -994. Thus this represents the starting point. Since -994 is a multiple of 2, it is eliminated; the next candidate is -987, which is a multiple of 3 and is also eliminated; then comes -980, which is a multiple of 2 again; the next one is -973, and it satisfies all the conditions.

   Strictly speaking, -973 is the only correct answer! However, since most solvers, as well as the problem posers, read "positive" into the problem, we also allowed 119 as a correct answer.

2. In the vicinity of the south pole all east-west travel is on a circle centred about the south pole. Since we wish to have the circumference of such a circle equal to 1 km, we must have the radius equal to $1/2\pi$ km. The original point from which the trip starts must be located a further 1 km away from the south pole. Thus we must start $1 + (1/2\pi)$ km from the south pole.

3. Since the number of apples is twice the number of bananas in each bowl as well as in the doctor's dictum, we can ignore the apple constraint, and simply solve the problem for bananas and pears. Since we have in each bowl either 0 or 3 pears, we see that the condition on the pears can be met in exactly one of 3 ways: 2 of bowl B and none of bowl C; 1 of each of bowls B and C; or none of bowl B and 2 of bowl C. In each case we can then add the number of A bowls to fill out the requirements. So there are three solutions: $(A, B, C) = (4, 2, 0)$, $(5, 1, 1)$, and $(6, 0, 2)$.

4. Let $\angle B = \angle C = x$. Let $\angle CDE = y$. Since $\angle AED$ is an exterior angle to $\triangle EDC$ we have $\angle AED = x + y$. Since $\triangle ADE$ is isosceles, we also have $\angle ADE = x + y$, whence $\angle ADC = x + 2y$. But $\angle ADC$ is an exterior angle of $\triangle ABD$, which means that $\angle ADC = x + \alpha$. Thus we have $x + 2y = x + \alpha$, or $y = \frac{1}{2}\alpha$. So $\angle CDE = \frac{1}{2}\alpha$.

5. Since $P^2 = P + 10k$ for some integer k, $0 \leq k \leq 8$, we see that P is one of 0, 1, 5, or 6. Now by considering the last 2 digits of each factor and the product we have $(10E+P)^2 = 100n+10E+P$ for some integer $n < 100$. This means that $20PE+P^2-10E-P = 10E(2P-1)+P(P-1)$ is a multiple of 100. Let us consider $P = 6$. Then $110E + 30$ is a multiple of 100, implying that $E = 7$. This means that 776^2 must end in the digits 776, but 776^2 actually ends in the digits 176. So $P \neq 6$. Next try $P = 5$. Then $90E + 20$ is a multiple of 100, implying

that $E = 2$. This means that 225^2 must end in the digits 225, but 225^2 actually ends in the digits 625. Thus $P \neq 5$. Therefore, $P = 0$ or 1. In either case we have $P(P-1) = 0$, which means that $10E(2P-1)$ is a multiple of 100. Since $2P - 1 = \pm 1$, we conclude that $E = 0$, implying that $P = 1$, since it must be different from E. So we have $P = 1$ and $E = 0$. Then we have $B = J^2$ and $B = 2J$, since $(J001)^2 = (J)^2 00(2J)001 = B00B001$. Since $J^2 = 2J$ and $J \neq 0$, we conclude that $J = 2$, whence $JEEP = 2001$.

Senior Final - Part A

1. The difference of squares $x^2 - y^2$ factors into $(x-y)(x+y)$. Since x and y are positive integers, we know that $x > y$. Thus $2001 = (x-y)(x+y)$ for four different sets of integers x and y. This requires that we determine how 2001 factors. With a little effort we see that $2001 = 3 \times 23 \times 29$. Thus the factorizations of 2001 are 1×2001, 3×667, 23×87, and 29×69. For $2001 = a \times b$ with $a < b$, we have $x - y = a$ and $x + y = b$, which means that $x = \frac{1}{2}(a+b)$. Thus for our four factorizations of 2001 we have $x = \frac{1}{2}(1 + 2001) = 1001$, $\frac{1}{2}(3 + 667) = 335$, $\frac{1}{2}(23 + 87) = 55$, and $\frac{1}{2}(29 + 69) = 49$, respectively. So the sum of the these four values is $1001 + 335 + 55 + 49 = 1440$.

   In the event that a student can only find the prime factor 3 for 2001, the problem can still be solved. As above we find that for the factorizations 1×2001 and 3×667, the values of x are 1001 and 335, respectively. Noting that these values are decreasing, we can see that no two additional values can be added to get as high as 2880, so 1440 is the only choice.

   Answer is (**d**)

2. Let x and y be the number of pears and peaches respectively that Antonino purchases. Then (in cents) he spends $18x + 33y = 2001$, which simplifies to $6x + 11y = 667$ or $6x = 667 - 11y$. Clearly the maximum number of fruits he could buy occurs when he maximizes the number of peaches (since they are cheaper), which means he should buy as few pears as possible. So let us try successive small values of y, starting at $y = 0$ to determine when $667 - 11y$ is first a multiple of 6. For $y = 0$, 1, 2, 3, 4, and 5 we get $667 - 11y = 667$, 656, 645, 634, 623, and 612. It is easy to check that 612 is the first of these which is a multiple of 6. Thus $y = 5$ and $x = 612/6 = 102$ Thus $x + y = 107$.

   Alternate approach: As above we have $6x + 11y = 667$. Let t be the total number of fruits purchased. Then $t = x + y$. Thus $6x + 6y = 6t$. Subtracting this from the first equation we have $5y = 667 - 6t$. So $667 - 6t$ must be a multiple of 5. Checking $t = 110$, we don't get a solution, but with $t = 107$ we have $y = 5$. Since we are interested in the largest total, our solution is 107.

   Answer is (**b**)

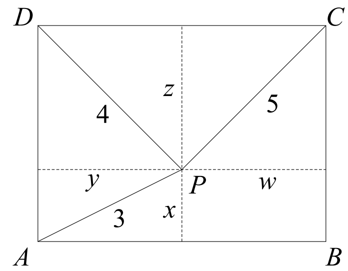
3. Set $x = \sqrt{3 + 2\sqrt{2}} - \sqrt{3 - 2\sqrt{2}}$. Then

$$\begin{aligned} x^2 &= 3 + 2\sqrt{2} - 2\sqrt{\left(3 + 2\sqrt{2}\right)\left(3 - 2\sqrt{2}\right)} + 3 - 2\sqrt{2} \\ &= 6 - 2\sqrt{3^2 - (2\sqrt{2})^2} = 6 - 2\sqrt{9 - 8} = 6 - 2 = 4 \end{aligned}$$

   Thus $x = \pm 2$. However, it is clear from the definition of x that it is positive, since $3 + 2\sqrt{2} > 3 - 2\sqrt{2}$. Therefore, $x = 2$.

   Answer is (**b**)

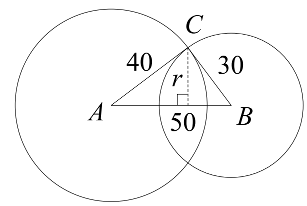
4. Draw lines through P parallel to the sides of the rectangle $ABCD$, cutting off lengths x, y, z, w, as shown in the diagram to the right. Then from the Theorem of Pythagoras we have



$$x^2 + y^2 = 3^2 = 9 \qquad (1)$$
$$y^2 + z^2 = 4^2 = 16 \qquad (2)$$
$$z^2 + w^2 = 5^2 = 25 \qquad (3)$$

If we now subtract (2) from the sum of (1) and (3) we get: $x^2 + w^2 = 18$. But we also have (from the Theorem of Pythagoras): $x^2 + w^2 = \overline{PB}^2$, whence PB has length $\sqrt{18} = 3\sqrt{2}$.
Answer is (**b**)

5. Consider a cross-section through the centres of the two spheres as shown in the diagram at the right. Let A be the centre of the sphere of radius 40 mm, and let B be the centre of the sphere of radius 30 mm. Let C be one of the points in this cross-section which lie where the two spheres join. Since the distance AB is 50 mm, we see by the Theorem of Pythagoras that $\triangle ABC$ is right angled with the right angle at point C. The altitude of this triangle is clearly the radius of the circle of intersection of the two bubbles. Let us denote this altitude by r. Then the area (in mm^2) of $\triangle ABC$ can be computed in 2 different ways:



$$A = \tfrac{1}{2}{\cdot}30{\cdot}40 = \tfrac{1}{2}{\cdot}50{\cdot}r$$

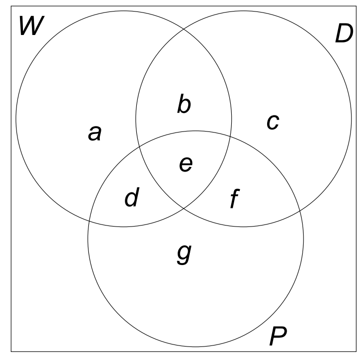
from which we see that $r = 24$ mm. Thus the diameter of the circle of intersection of the spheres is 48 mm.
Answer is (**b**)

6. If we let those people in line possessing only a toonie be denoted by T, and those possessing a loonie be denoted by L, then our problem can be translated to: what is the probability of a random list of four Ls and four Ts having the property that in moving from the beginning of the list to the end of the list we will have always encountered at least as many Ls as Ts. To begin we will first determine the total number of possible random orderings of four Ls and four Ts. Clearly there are 8 positions in the list, 4 of which must be set aside for L, with the remainder having T. This means there are in total $\binom{8}{4} = 70$ such random orderings of four Ls and four Ts. Now let us try to determine the number of such orderings satisfying the additional condition that in moving from the beginning of the list to the end of the list we always encounter at least as many Ls as Ts. Let us examine the first 4 positions in the list. We note that there must be at least 2 Ls in these first 4 positions. We also note that however many Ls there are among the first 4 positions, there are the same number of Ts among the last 4 positions of the list.
Case (i): there are 4 Ls among the first 4 positions. In this case there is only one possibility, namely $LLLLTTTT$.
Case (ii): there are 3 Ls among the first 4 positions. Since the first position must be L, there are 3 places where one can put the T that belongs to the first 4 positions. Thus there are 3 possible arrangements for the first 4 positions. But by symmetry, there are also 3 possible arrangements for the last 4 positions, and the first 4 positions and the last 4 positions can be arranged independently, for a total of $3 \times 3 = 9$ possibilities for case (ii).
Case (iii): there are 2 Ls among the first 4 positions. Again the first position must be L. It is easy to see that there are only 2 possible arrangements among the first 4 positions, namely

$LLTT$ and $LTLT$. By symmetry, we have the same number of possibilities for the last 4 positions, for a total of $2 \times 2 = 4$ arrangements for case (iii).

Thus we have a total of $1+9+4 = 14$ acceptable arrangements, and the probability we seek is $14/70 = 1/5$. <u>Answer is (**d**)</u>

7. Consider the diagram to the right, where the three circles represent the applicants with design skills (D), writing skills (W), and programming skills (P). We have used the letters a through g to represent the various subsets of these people having different combinations (or lack) of skills. We are interested in the value of e. Since 80% of the 45 applicants have at least one of the desired skills, there are 36 such applicants. From the remaining information in the problem statement we conclude that



$$\begin{array}{rcl}
a + b + c + d + e + f + g &=& 36 \\
b + c \quad\;\; + e + f \quad\;\; &=& 20 \\
a + b \quad\;\; + d + e \qquad\quad &=& 25 \\
d + e + f + g &=& 21 \\
b \qquad\quad + e \qquad\quad &=& 12 \\
d + e \qquad\quad &=& 14 \\
e + f \quad\;\; &=& 11
\end{array}$$

Adding the second, third, and fourth equations above and subtracting the first we get $b + d + 2e + f = 30$, while adding the last three equations together yields $b + d + 3e + f = 37$. Comparing these we see that $e = 7$. This is all we need to answer the question. However, the interested reader may be curious to find all the remaining values as well, so we continue. With this value of e we can use the last three equations displayed above to determine $b = 5$, $d = 7$, and $f = 4$. With these values we can use the second, third, and fourth equations displayed above to determine $c = 4$, $a = 6$, and $g = 3$. One can simply check that the first equation is satisfied for these values. <u>Answer is (**b**)</u>

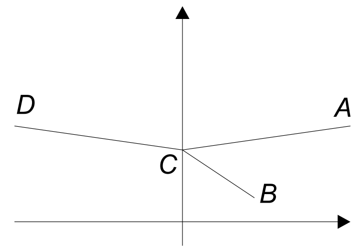
8. Let us label the critical equation:

$$a \textcircled{S} (b \textcircled{S} c) = (a \textcircled{S} b) \textcircled{L} (a \textcircled{S} c), \tag{1}$$

where we are assuming that a, b, c are three distinct numbers. Clearly, the left hand side of this expression is always the smallest of the three values a, b, c. If the smallest of the three values is b, then the left hand side of (1) is b, while the right hand side simplifies to $b \textcircled{L} (a \textcircled{S} c)$, which is definitely a or c; thus b cannot be the smallest of the three values. Similarly, c cannot be the smallest of the three values. This means that a is the smallest values. This eliminates all but choice (a) and choice (e) from the set of possible answers. Now, if we examine (1) with a the smallest value, then both sides resolve to a, and we have (1) holding true. This describes choice (a). We note that choice (e) imposes a further restriction, namely $b < c$, which is unnecessary. We are asked to determine which <u>must</u> hold. Thus, our solution is simply that a must be the smallest of the three values.

<u>Answer is (**a**)</u>

9. Consider also the point D with coordinates $(-7, 4)$ (see diagram at the right). Clearly $\overline{AC} = \overline{DC}$. Thus we must find k which minimizes the sum $\overline{DC} + \overline{BC}$. This sum is obviously minimized when B, C, and D are collinear (i.e. when they lie on one line). This occurs when the slope of BC is equal to the slope of DC. The slope of DC is $-\frac{3}{10}$ and the slope of BC is $(1-k)/3$. Setting these equal yields $1 - k = -\frac{9}{10}$, or $k = 1.9$. Answer is (**c**)



10. Observe that the unshaded portion of the quarter circle is also $\frac{1}{2}$ of its area. Let us then compute the area of the unshaded regions. We will solve the more general problem using a radius of r units. Clearly, the area of triangle CBX is $\frac{1}{2}xr$. Now drop a perpendicular from A to the line CD meeting it at E. Since $\angle ACD = 30°$, we see that $\overline{AE} = \frac{1}{2}r$, and by the Theorem of Pythagoras we then get $\overline{CE} = \frac{\sqrt{3}}{2}r$. Thus the area of triangle AXE is $\frac{1}{2}\left(\frac{\sqrt{3}}{2}r - x\right) \cdot \frac{1}{2}r$. The remaining unshaded region is the curved piece AED. This is obviously the difference between the circular sector ACD and triangle ACE. The sector ACD has area $\frac{1}{12}\pi r^2$, since it is one twelfth part of a circle of radius r. The triangle ACE has area $\frac{1}{2} \cdot \frac{\sqrt{3}}{2} r \cdot \frac{1}{2} r$. Putting all the pieces together we see that the area of the unshaded part is:

$$A = \frac{xr}{2} + \left(\frac{\sqrt{3}r^2}{8} - \frac{xr}{4}\right) + \left(\frac{\pi r^2}{12} - \frac{\sqrt{3}r^2}{8}\right) = \frac{xr}{4} + \frac{\pi r^2}{12}$$

But we are given that A is one half the area of the quarter circle, i.e. one half of $\frac{1}{4}\pi r^2$. Thus we have

$$\begin{aligned} \frac{xr}{4} + \frac{\pi r^2}{12} &= \frac{\pi r^2}{8} \\ \frac{xr}{4} &= \frac{\pi r^2}{24} \\ x &= \frac{\pi r}{6} \end{aligned}$$

Since $r = 1$ we have $x = \pi/6$.

Alternate approach: Note that the shaded area can be computed by subtracting from the area of the quarter circle both the area of $\triangle CBX$ and the difference between the areas of the circular sector ACD and $\triangle CAX$. So the area we seek is:

$$\begin{aligned} &\tfrac{1}{8}\pi r^2 = A = \frac{1}{4}\pi r^2 - \frac{1}{2}rx - \left[\frac{1}{12}\pi r^2 - \frac{1}{2}x\left(\frac{1}{2}r\right)\right] = \frac{1}{6}\pi r^2 - \frac{1}{4}rx \\ &\therefore \quad \frac{1}{4}rx = \frac{1}{24}\pi r^2 \\ &\text{or} \qquad x = \frac{1}{6}\pi r \end{aligned}$$

Again with $r = 1$ we get $x = \pi/6$. Answer is (**c**)

Senior Final - Part B

1. This problem is the same as #4 on the Junior Final (Part B).

2. (The reader may be interested to note that this is a generalization of #13 on the Senior Preliminary.) To obtain the area of the shaded region we will compute the sum of the areas of triangle ABC, the area of the semicircle on AB, and the area of the semicircle on AC, and then we will subtract from this sum the area of the semicircle on BC. Let a, b, and c be the lengths of the sides BC, AC, and AB, respectively. Since $\triangle ABC$ is inscribed in a semicircle on BC, we see that $\angle BAC = 90°$. Thus by the Theorem of Pythagoras we have $a^2 = b^2 + c^2$. If we now denote the area of $\triangle ABC$ by $[ABC]$, then our desired area is:

$$\begin{aligned}\mathcal{A} &= [ABC] + \tfrac{1}{2}\pi\left(\tfrac{c}{2}\right)^2 + \tfrac{1}{2}\pi\left(\tfrac{b}{2}\right)^2 - \tfrac{1}{2}\pi\left(\tfrac{a}{2}\right)^2 \\ &= [ABC] + \tfrac{1}{8}\pi(b^2 + c^2 - a^2) = [ABC]\end{aligned}$$

3. Since each school had exactly one entry in each event, we conclude by (i) that Cranbrook had 4 first place finishes and 1 second place finish. By (ii) it becomes clear that the 1 second place finish they had was in the high jump. Thus Doug Dolan of Duchess Park could finish no higher than third place in the high jump. Of the total of 75 available points, 24 went to Cranbrook, which leaves 51 points to be shared by the other 4 schools. Since they all received different totals, Duchess Park, who came in second must have obtained at least 15 points (since $14 + 13 + 12 + 11 = 50$, which is too small). A similar argument shows that last place Selkirk must have obtained at most 11 points (since $12 + 13 + 14 + 15 = 54$, which is too large). Since Selkirk obtained 5 points for the high jump and 3 points for the pole vault by (ii), and at least 1 point for each of the other 3 events, they must have a total of at least 11. This, together with our previous remark shows that Selkirk had exactly 11 points. This leaves only 40 points to be shared by Duchess Park, Nanaimo, and Okanagan Mission, and each of them must have at least 12 points. The only possibility is that Duchess Park had 15 points, Nanaimo had 13 points and Okanagan Mission had 12 points. Since Nanaimo received the same number of points in 4 of the five events and had a total of 13 points, they must have finished third four times and last once (since four second place finishes would give them too many points, while four fourth place finishes would require them to finish first in the other event to get 13 points, but all the first place finishes went to Cranbrook and Selkirk). Thus Nanaimo had to finish last in the pole vault, as Selkirk finished third. At this point we have determined that all 1-point, 3-point, and 5-point finishes (except for last place in the high jump) have gone to one of Cranbrook, Nanaimo, or Selkirk. Since the only remaining odd point will generate an odd total, it must go to Duchess Park, which has a total of 15 points. Thus Doug Dolan of Duchess Park must have finished last in the high jump.

4. Let g be the number of green tickets in the box. Then the total number of tickets in the box is $g + 3$. The number of ways of drawing two tickets from the box (together) is $\binom{g+3}{2} = (g + 3)(g + 2)/2$. The number of ways of drawing 1 ticket of each colour is by drawing 1 of 3 blue tickets and 1 of g green tickets, which is $3 \cdot g$. Thus the probability of drawing 1 of each colour when drawing two tickets together is

$$\frac{3g}{(g+3)(g+2)/2} = \frac{6g}{(g+3)(g+2)}$$

We are told that this probability is $\frac{1}{2}$. Therefore, we have

$$\begin{aligned} \frac{1}{2} &= \frac{6g}{g^2+5g+6} \\ g^2+5g+6 &= 12g \\ g^2-7g+6 &= 0 \\ (g-6)(g-1) &= 0 \end{aligned}$$

which means that $g = 1$ or $g = 6$. Both of these solutions can be verified.

5. (a) Note that

$$100h + 10t + u = 99h + 11t + (h - t + u) = 11(9h + t) + (h - t + u)$$

Clearly, if $h - t + u$ is divisible by 11, the entire right hand side is also divisible by 11, which means that $100h + 10t + u$ is divisible by 11.

(b) In this case we observe

$$\begin{aligned} 1000m + 100h + 10t + u &= 1001m + 99h + 11t + (-m + h - t + u) \\ &= 11(91m + 9h + t) + (h + u - m - t) \end{aligned} \tag{1}$$

Again, if $h + u = m + t$ we see that the right hand side is simply $11(91m + 9h + t)$, which is clearly divisible by 11, implying that the right hand is also divisible by 11.

(c) If we examine (1) in part (b) above, we see that to have the right hand side divisible by 11 all we need is that $h + u - m - t$ is divisible by 11. This can happen without $h + u = m + t$ as the following example shows: $h = 9$, $u = 8$, $m = 4$, and $t = 2$; $h + u = 17$ and $m + t = 6$, which means that $h + u \neq m + t$, but $h + u - m - t = 11$ and the expression in (1) above is then divisible by 11.