BRITISH COLUMBIA COLLEGES
High School Mathematics Contest
2004 Solutions

Junior Preliminary

1. Let U, H, and F denote, respectively, the set of 2004 students, the subset of those wearing Hip jeans, and the subset of those wearing Fast runners. For S a subset of U, let S^c denote the complement of S in U, and let $n(S)$ denote the number of elements in S. Then we must find $n(H^c \cap F^c)$.

   Now, the subsets $H \cap F$, $H^c \cap F$, $H \cap F^c$, and $H^c \cap F^c$ are pair-wise disjoint, and their union is all of U (i.e., they *partition* U). Then

   $$U = (H \cap F) \cup (H^c \cap F) \cup (H \cap F^c) \cup (H^c \cap F^c)$$

   so

   $$n(U) = n(H \cap F) + n(H^c \cap F) + n(H \cap F^c) + n(H^c \cap F^c)$$

   and hence

   $$n(H^c \cap F^c) = 2004 - 239 - 253 - 1013 = 300.$$

   answer is (c).

2. Kate requires three cuts for the first log and four cuts for the second log. The time per cut for the first log is 9 seconds / 3 cuts = 3 seconds per cut. Since this time remains constant for the second log, Kate requires (3 sec/cut) x 4 cuts = 12 seconds to cut the second log into five pieces.

   answer is (a).

3. Denote $(0,0)$, $(1,5)$, and $(7,3)$, by O, P, and Q, respectively, and for R and S any points in the plane, let $d(R,S)$ denote the distance from R to S. Then $d(O,P) = \sqrt{26}$, $d(P,Q) = \sqrt{40}$, and $d(O,Q) = \sqrt{58}$. Since OQ is the longest side, the perpendicular from P to the line passing through O and Q intersects OQ at a point R (say) between O and Q. Let $d(O,R) = a$ and let $d(R,Q) = b$. Then $d(R,Q) = \sqrt{58} - a$, and so, by Pythagoras' Theorem, $a^2 + b^2 = 26$ and $(\sqrt{58} - a)^2 + b^2 = 40$. Solving for a gives $a = 11\sqrt{58}/29$. Then $b = \sqrt{26 - (11\sqrt{58}/29)^2} = \sqrt{512/29}$ so the area of triangle OPQ is $(1/2)\sqrt{58}\sqrt{512/29} = 16$.

   Alternate solution: We can use Heron's formula which gives the area A of a triangle intrinsically in terms of the lengths of the three sides a, b, and c of the triangle. Let $s = (a + b + c)/2$ be the semi-perimeter of the triangle. Then Heron's formula states $A = \sqrt{s(s-a)(s-b)(s-c)}$. Now, from the initial calculations in the first solution above, we compute directly A = 16.

   answer is (b).

4. Let r be the number of red pencils and let b be the number of blue pencils. Then $r = \dfrac{2}{3}b$ or $b = \dfrac{3}{2}r$. Then the proportion of red pencils is $\dfrac{r}{r+b} = \dfrac{r}{r + \frac{3}{2}r} = \dfrac{2}{5}$.

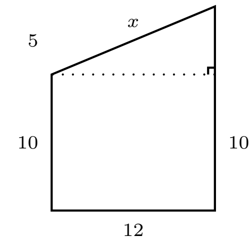
   answer is (d).

5. The increase in the cost of a skateboard is \$100(0.12) = \$12 and the increase in the cost of a helmet is \$40(0.05) = \$2 for an overall cost increase of \$14. So, the percent increase in overall cost is $\dfrac{\$14}{\$140} \times 100\% = 10\%$.

   answer is (b).

6. By Pythagoras' Theorem, the required distance x satisfies

$$x^2 = 12^2 + 5^2 = 169 \Rightarrow x = 13$$



answer is (d).

7. First note that no digit can be less than 7 since $34 - 7 = 27 = 3 \times 9$, the maximum sum for the other three digits. Second, note there must be at least two 9s since $34 - 9 = 25 > 3 \times 8$, the maximum sum for the other three digits. Clearly there is just one number with four nines. If there are three 9s, then the other digit can be 7 or 8. Since either of these can be in four possible places, there are eight numbers with three 9s. If there are two 9s, then the other two digits must be 8s. Since the placement of either the two 8s or the two 9s determines the places of the other pair, the number of ways we can arrange them in the number is the number of ways of placing two objects in four places (i.e., the number "combinations of 4 choose 2". There are $\frac{4\times 3}{2} = 6$ such combinations and thus a total of $1 + 8 + 6 = 15$ numbers meeting the conditions of the problem.

answer is (c).

8. Let P and A denote, respectively, the perimeter and area of the large square, and let p and a denote the same for the smaller square. Then $A = \left(\dfrac{P}{4}\right)^2 = 25$ and $a = \frac{25}{2}$, so $p = 4\sqrt{\dfrac{25}{2}} = 10\sqrt{2}$.

answer is (b).

9. Given $A \star B = \dfrac{A + 2B}{3}$, we have $4 \star 7 = \dfrac{4 + 14}{3} = \dfrac{18}{3} = 6$, so $(4 \star 7) \star 8 = 6 \star 8 = \dfrac{6 + 16}{3} = \dfrac{22}{3}$.

On the other hand, $7 \star 8 = \dfrac{7 + 16}{3} = \dfrac{23}{3}$, so $4 \star (7 \star 8) = 4 \star \dfrac{23}{3} = \dfrac{4 + \frac{46}{3}}{3} = \dfrac{58}{9}$.

Finally, $[(4 \star 7) \star 8] - [4 \star (7 \star 8)] = \dfrac{22}{3} - \dfrac{58}{9} = \dfrac{8}{9}$.

answer is (d).

10. Clearly we can select all 20 yellow and all 6 red discs without selecting at least 2 blue discs. This gives 29. Now if we select 2 blue discs we have a total of 31.

answer is (e).

11. Evidently $\dfrac{d + b + c}{a} = \dfrac{3}{4}$ and $\dfrac{b + c}{a} = \dfrac{5}{3}$. Then $\dfrac{d}{a} = \dfrac{d + b + c}{a} - \dfrac{b + c}{a} = \dfrac{3}{4} - \dfrac{5}{3} = -\dfrac{11}{12}$.

answer is (d).

12. (See Senior Preliminary Problem 5 for alternate solutions.) Let x denote the number of nickels Luke has. Then, by the division algorithm for integers, x is the smallest natural number satisfying the equations $x = 3r + 1 = 5s + 2 = 7t + 3$ for some natural numbers r, s, and t. Now $r \geq 0$ so (again by the division algorithm) $r = 5u + v$ for some natural numbers u and v, with $0 \leq v \leq 4$. Then $x = 3(5u + v) + 1 = 15u + 3v + 1 = 5s + 2$ so, by the fundamental theorem of arithmetic, 5 is a factor of $3v - 1$. Hence $v = 2$. Substituting this back in the left hand side we find $15u + 7 = 7t + 3$ (from above). As before, $u = 7a + b$ for some natural numbers a and b with $0 \leq b \leq 6$ so $15(7a + b) + 7 = 7t + 3$. Then 7 must be a factor of $15b - 3$ and hence also of $5b - 1$. Since $0 \leq b \leq 6$, this implies $b = 3$. Thus $x = 7(15)u + 7 + 15(3)$ and this is a minimum when $u = 0$ in which case $x = 52$.

answer is (b).

13. We have
$$(6^{30}+6^{-30})(6^{30}-6^{-30}) = 3^A8^B - 3^{-A}8^{-B}$$
so
$$6^{60}-6^{-60} = 3^A2^{3B} - 3^{-A}2^{-3B}$$
and
$$3^{60}2^{60} - 3^{-60}2^{-60} = 3^A2^{3B} - 3^{-A}2^{-3B}.$$
Whence $A = 60$, $B = 20$, and $A + B = 80$.

answer is (d).

14. Let $x = 0.7\overline{18}$ (that is, the digit block 18 repeats infinitely often). Then
$$\begin{array}{rrcl} & 1000x & = & 718.\overline{18} \\ & 10x & = & 7.\overline{18} \\ \text{Subtracting, we have} & 990x & = & 711 \end{array}$$
So $x = \dfrac{711}{990} = \dfrac{79}{110}$. So, when written in lowest terms, the denominator of x exceeds its numerator by $110 - 79 = 31$.

answer is (b).

15. (See Senior Preliminary Problem 1 for an alternate solution.) We use the *fundamental principle of counting* which states that if a certain task can be done in m ways and another task can be done in n ways, then the two tasks together can be done in mn ways. This principle was used in the solutions to problems 6. and 7. above. In the present case, we can think of the tasks as: ordering the subjects; and then ordering each of the three subjects. Thus there are four initial tasks. Now, there are three subjects and so $3\times2\times1 = 6$ ways we can order the subjects. Now we consider the tasks of ordering each subject. There are three mathematics books and they can be can be ordered in $3 \times 2 \times 1 = 6$ ways. Similarly, the English books can be ordered in $2 \times 1 = 2$ ways; and the science books in $4 \times 3 \times 2 \times 1 = 24$ ways. Therefore we can form $6 \times 6 \times 2 \times 24 = 1728$ different orderings of all of the books.

answer is (e).

Senior Preliminary

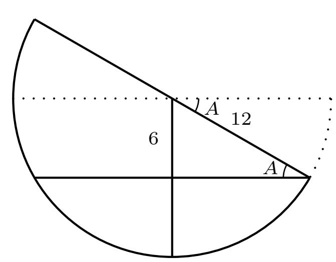
1. (See Junior Preliminary Problem 15 for an alternate solution.) First, there are $3 \cdot 2 \cdot 1 = 6$ ways to mix up the kinds of books. There are $3 \cdot 2 \cdot 1 = 6$ ways to mix up the math books, 2 ways to mix up the English books and $4 \cdot 3 \cdot 2 \cdot 1 = 24$ ways to mix up the science books. The total number of ways will be the product: $6 \cdot 6 \cdot 2 \cdot 24 = 1728$.

answer is (e).

2. This is the same as Junior Preliminary Problem 13.

answer is (d).

3. Take a cross-section of the tilted bowl. The tilt angle is the angle A. From the diagram,
$$\sin A = \frac{6}{12} = \frac{1}{2}$$
and
$$A = 30^\circ.$$



answer is (b).

4. The idea is to get the time taken to travel up the hill and the time taken to travel down. If D is the distance to the top of the hill, the time taken to travel down will be

$$4.5 = \frac{D}{\text{time}}$$

or

$$\text{time} = \frac{D}{4.5}.$$

The time taken to travel up will be

$$\frac{D}{1.5}.$$

The total time taken will be

$$\begin{aligned} \frac{D}{4.5} + \frac{D}{1.5} &= 6 \\ \frac{D + 3D}{4.5} &= 6 \\ 4D &= 27 \\ D &= \frac{27}{4} \end{aligned}$$

answer is (d).

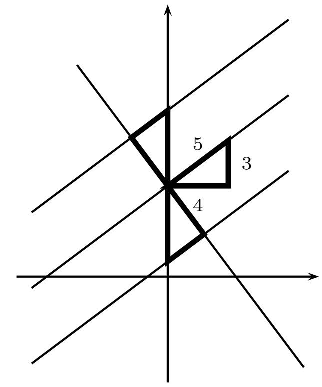
5. (See Junnior Preliminary Problem 12 for an alternate solution.) We use brute force, which works when the number of nickels is small. Let n be the number of nickels. When collected in groups of 3, there is one left over, so that n is a multiple of 3, plus 1. Similarly, n is a multiple of 5, plus 2, and also a multiple of 7, plus 3. Since multiples of 7 take bigger steps, we list possible n's and check whether they satisfy all requirements in the first three columns of the table below:

multiple of 7 plus 3	multiple of 5 plus 2	multiple of 3 plus 1	remainder on division by 5	remainder on division by 3
3	×	×	3	0
10	×	OK	0	1
17	OK	×	2	2
24	×	×	4	0
31	×	OK	1	1
38	×	×	3	2
45	×	×	0	0
52	OK	OK	2	1

52 works and the sum of its digits is 7

Alternate Solution: We look at the remainders of multiples of 7, plus 3 after division by 5 and 3 in the last two columns of the table above. The remainders after division by 5 repeat every 5, remainders after division by 3 repeat every 3. Since 17 is the first number that has a remainder of 2 after division by 5, the number we seek must be of the form $17 + 7 \cdot 5 \cdot n$ for some n. The first number that has a remainder of 1 after division by 3 is 10, so the number is also of the form $10 + 7 \cdot 3 \cdot m$. Subtracting, $7 + 35n - 21m = 0$, or $1 = 3m - 5n$. The first pair that works is $m = 2$ and $n = 1$, which gives 52.

answer is (b).



6. The distance between parallel lines is measured along a line perpendicular to both parallel lines. That line has slope $-\frac{4}{3}$. A point on the original line is $(0,6)$, so the equation of the perpendicular line is $y=-\frac{4}{3}x+6$. A point 4 units from $(0,6)$ will have x-coordinates given by

$$\sqrt{(x-0)^2+\left[\left(-\frac{4}{3}x+6\right)-6\right]^2}=4$$

$$x^2+\frac{16}{9}x^2=16 \Rightarrow \frac{25}{9}x^2=16 \Rightarrow x^2=\frac{144}{25} \Rightarrow x=\pm\frac{12}{5}$$

Then, the corresponding y-coordinates are

$$y=-\frac{4}{3}\left(\pm\frac{12}{5}\right)+6=\pm\frac{16}{5}+6=\frac{46}{5},-\frac{14}{5}$$

The two lines are

$$y-\frac{46}{5}=\frac{3}{4}\left(x+\frac{12}{5}\right)=\frac{3}{4}x+\frac{9}{5} \Rightarrow y=\frac{3}{4}x+\frac{55}{5}=\frac{3}{4}x+11$$

and

$$y-\frac{14}{5}=\frac{3}{4}\left(x-\frac{12}{5}\right)=\frac{3}{4}x-\frac{9}{5} \Rightarrow y=\frac{3}{4}x+1$$

Alternate solution: The three triangles in the diagram are congruent, since they have the same angles and the perpendicular distance between the lines, 4, is the longer of the two perpendicular sides of the triangles along the y-axis. The triangle on the right defines the slope of the line, giving the sides as labelled, with the longer of the two perpendicular sides being 4. Thus, the other two triangles have hypotenuse 5, which is the vertical distance between the lines. Hence, the y-intercepts of the two lines 4 units from the given line are $b=6-5=1$ and $b=6+5=11$. Thus, the equations of the two lines are $y=\frac{3}{4}x+11$ and $y=\frac{3}{4}x+1$.

answer is (b).

7. We add up the distances walked. The first leg is 19.5 metres long (we turn half a metre from the end so that we walk down the centre of the next leg), the next leg is 9 m long, then 19 m, 8 m, 18 m, 7 m, 17 m, 6 m, 16 m, 5 m, 15 m, 4 m, 14 m, 3 m, 13 m, 2 m, 12 m, 1 m and 11.5 m. The total length is 200 m.

Alternate solution: The numbers almost pair off: listing them "snake fashion",

19.5	9	19	8	18	7	17	6	16
1	12	2	13	3	14	4	15	5
11.5								

all except the first "column" add to 21. We have $8\cdot 21+32=200$ m.

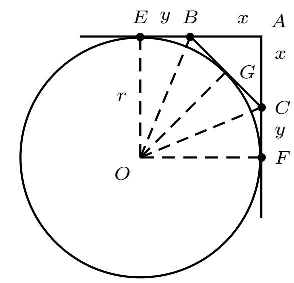
Alternate solution: For a strip of paper 1 metre wide the length and area are the same. If the strip has a dotted line down the centre, is cut and the two pieces placed at right angles so that the dotted lines join up, then the pieces of paper overlap in a square of side 0.5m. The area of an L-shaped piece of paper 1 metre wide is therefore the same as the length of the path down the centre of the paper. For our maze, the length of the path is the same as the area, and is $10\times 20=200$ metres.

answer is (b).

8. If the disc cannot touch a grid line, it must lie entirely inside of one of the grid squares. This is the case if the centre of the disc lies inside a $2\,\text{cm} \times 2\,\text{cm}$ square. The area of this square is $4\,\text{cm}^2$ and the area of a grid square is $25\,\text{cm}^2$. So the probability that the disc does not touch a grid line is $\frac{4}{25} = 0.16$.

answer is (e).

9. In the diagram, O is the centre of the circle, G is the midpoint of the line segment BC, and line segments OE, OB, OG, OC, and OF have been added. Further, let $x = \overline{AB} = \overline{BC}$ and $y = \overline{BE} = \overline{CF}$. Then, since triangle ABC is an isosceles right triangle with area 9, $\frac{1}{2}x^2 = 9$, so that $x = 3\sqrt{2}$. Further, triangles BEO and BGO are right triangles with equal sides, so that they are congruent. Thus, $BG = CG = y$ and $BC = 2y$. Then, using Pythagoras' on triangle ABC gives $x^2 + x^2 = 2x^2 = (2y)^2$, so that $y^2 = 9$ and $y = 3$. Finally, the radius of the circle is $r = x + y = 3\sqrt{2} + 3 = 3\left(\sqrt{2}+1\right)$, so that the area is $\pi r^2 = 9\pi\left(\sqrt{2}+1\right)^2 = 9\pi\left(3 + 2\sqrt{2}\right)$.



answer is (c).

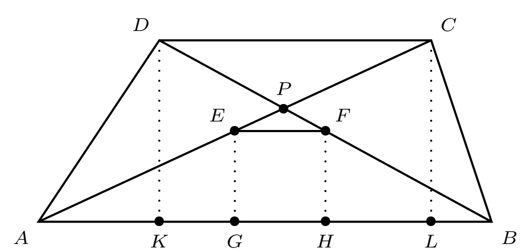
10. First notice that the shaded region is a square since if you rotate the diagram by 90°, it does not change. To find its area, notice that the area of the triangle joining the upper right-hand corner and the midpoint of the side opposite, plus the area of the triangle joining the lower left-hand corner and the midpoint of the side opposite is half the area of the large square. The diagonal parallelogram therefore has area $\frac{1}{2}$. The length of the line joining one of the corners to the midpoint of the side opposite has length $1 + \left(\frac{1}{2}\right)^2 = \frac{\sqrt{5}}{2}$. If h is the height of the parallelogram, and therefore the length of one side of the square, then

$$\begin{aligned} \text{Area parallelogram} &= \text{length} \times \text{height} \\ &= \frac{\sqrt{5}}{2}h \\ &= \frac{1}{2} \end{aligned}$$

so that $h = \frac{1}{\sqrt{5}}$, and the area of the square is $\frac{1}{5}$.

answer is (e).

11. In the diagram E and F are the midpoints of the diagonals AC and BD. Let h be the perpendicular distance between the sides AB and CD of the trapezoid. This is the height of the trapezoid. Let h_1 be the height of the triangle EFP measured from vertex P to side EF, h_2 the height of the triangle ABP measured from vertex P to side AB, and h_3 the height of the triangle CDP measured from vertex P to side CD. From the given information we have $\overline{AB} = 15$ and $\overline{EF} = 3$, so by similar triangles EFP and ABP



$$\frac{h_1}{h_2} = \frac{\overline{EF}}{\overline{AB}} = \frac{1}{5}$$

But, since E and F are the midpoints of the diagonals, by similar triangles ACL and AEG

$$h_2 - h_1 = \tfrac{1}{2}h \Rightarrow \tfrac{4}{5}h_2 = \tfrac{1}{2}h \Rightarrow h_2 = \tfrac{5}{8}h$$

Thus,

$$h_3 = h - h_2 = \tfrac{3}{8}h$$

Finally, by similar triangles CDP and ABP

$$\frac{\overline{DC}}{\overline{AB}} = \frac{h_3}{h_2} = \frac{\left(\frac{3}{8}h\right)}{\left(\frac{5}{8}h\right)} = \frac{3}{5} \Rightarrow \overline{DC} = \frac{3}{5}\overline{AB} = 9$$

Alternate solution: From the diagram we see that

$$\overline{AG} + \overline{BH} = \overline{AB} - \overline{EF} = 12$$

Now, by similar triangles $\overline{AG} = \frac{1}{2}\overline{AL}$ and $\overline{BH} = \frac{1}{2}\overline{BK}$, so that

$$\overline{AL} + \overline{BK} = 24$$

Further,

$$\begin{aligned} \overline{AL} + \overline{BK} &= \overline{AB} - \overline{BL} + \overline{AB} - \overline{AK} \\ &= \overline{AB} + \overline{AB} - \left(\overline{BL} + \overline{AK}\right) = \overline{AB} + \overline{DC} \end{aligned}$$

Hence,

$$\overline{DC} = \left(\overline{AK} + \overline{BK}\right) - \overline{AB} = 9$$

answer is (c).

12. Join E and C. The region is then a triangle and a trapezoid. If we know the length of EC and the height of the trapezoid, we can find the total area. dropping a vertical from E and extending the line AB to the vertical, we have a right-angle triangle whose hypotenuse is 2 units long. The angle at C is $60°$, so the vertical has length $2\sin 60° = \sqrt{3}$. The missing side has length 1 unit, and EC, by symmetry, has length 4 units. The area of the trapezoid is $\frac{1}{2}(2+4)\sqrt{3} = 3\sqrt{3}$. The triangle is equilateral with all sides of length 4. Dropping a vertical from D, the height of the triangle is $4\frac{\sqrt{3}}{2} = 2\sqrt{3}$. The area of the triangle is $\frac{1}{2}\left(2\sqrt{3}\right)(4) = 4\sqrt{3}$. The area of the whole figure is $7\sqrt{3}$.

answer is (a).

13. For the largest group to be of smallest possible size, all groups must be of roughly the same size. Otherwise, the largest group could be reduced by one, and the smallest increased by one. Since everyone can have between no and 20 bills, there are 21 possibilities. $\dfrac{600}{21} \sim 28.57$. The smallest largest group will be 29 (not 28 since $21 \times 28 < 600$).

answer is (d).

14. If the original number is $1a$, then when the 1 is moved to the end, the new number is a1. Since $1a$ is really $1 \times 10^n + a$ for some n, and $a1$ is $10a + 1$,

$$\begin{aligned} 3\left(10^n + a\right) &= 10a + 1 \\ 3 \times 10^n + 3a &= 10a + 1 \\ 3 \times 10^n - 1 &= 7a \end{aligned}$$

$3 \times 10^n - 1$ is of the form $2999\ldots9$. The first of these divisible by 7 will give us a:

$$\begin{array}{r@{\,}l} & 42857 \\ 7 & \overline{\big)\,29999\ldots} \\ & \underline{28} \\ & 19 \\ & \underline{14} \\ & 59 \\ & \underline{56} \\ & 39 \\ & \underline{35} \\ & 49 \\ & 49 \end{array}$$

$a = 42857$, the number is 142857. The sum of the digits is divisible by 9, so that the remainder on division by 9 is zero.

answer is (a).

15. Since the base is square and each triangle is equilateral, all sides have the same length, which we will call s. If we drop a vertical from A, it will bisect the line BD at F. The length of the line BD is $\sqrt{2}s$. The right-angle triangle ABF has hypotenuse s and the side adjacent to B has length $\frac{\sqrt{2}}{2}s$. The angle ABD must be a $45°$ angle.

answer is (d).

Junior Final Part A

1. Let a, b, and c denote the lengths of the three sides; c the hypotenuse. Then, by Pythagoras' Theorem, $a^2 + b^2 + c^2 = 10$ and $a^2 + b^2 = c^2$. Hence $2c^2 = 10$ so $c = \sqrt{5}$ (c being a length).

answer is (a).

2. Let d, t, and v denote, respectively, distance, time and velocity. We use the subscript 1 for the outward journey, 2 for the homeward journey, and 3 for the complete journey. Then $d_1 = 80t_1$, and $d_2 = 100t_2$, so $t_2 = 4/5t_1$, and hence $v_3 = \dfrac{d_1 + d_2}{t_1 + t_2} = \dfrac{80t_1 + 100t_2}{t_1 + t_2} = 80 + \dfrac{20t_2}{t_1 + t_2} = 80 + \dfrac{20(4t_1/5)}{t_1 + 4t_1/5} = 80 + \frac{80}{9} = 89 - \frac{1}{9}$ which is 89 to the nearest integer.

answer is (e).

3. Since the digits of an increasing number increase from left to right, they must be distinct. Consider the task of choosing four digits from the five digits 5, 6, 7, 8, and 9. Since choosing four of them leaves only one behind, the number of ways of choosing four of them is the same as the number of ways of choosing one of them, and, clearly, this number is five. Now, once four digits have been chosen, there is only one way of arranging them in increasing order. Hence there are five increasing numbers between 5000 and 10000.

   Alternate solution: Since the digits must be distinct, intuition suggests there will not be that many increasing numbers between 5000 and 10000. Hence we could try to enumerate them explicitly. Consideration of the possible digits in each place then produces the five numbers: 5678, 5679, 5689, 5789, and 6789. (The advantage of the first solution is that it works in cases where the explicit enumeration of all increasing numbers would be prohibitive).

answer is (c).

4. Note that, for $n \geq 5$, $n!$ is divisible by 10 and so has units digit 0. Therefore, the units digit of $S = 1! + 2! + 3! + ... + 50!$ is the same as the units digit of $1! + 2! + 3! + 4! = 33$. So the answer is 3.

answer is (c).

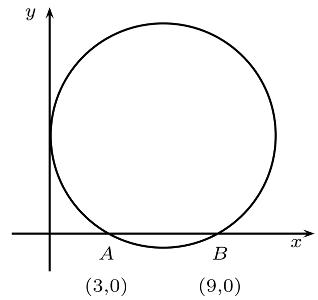
5. Not all of the clues are relevant to the third column. The first clue in the second line, "♡ and ○ are not adjacent", requires that □ is the third symbol in the third column. The third clue in the first line, "D is beside C", requires that either C or D is the fourth symbol in the third column and that the other letter is fourth symbol in the fourth column. The last clue requires that D is the fourth symbol in the third column.

answer is (a).

6. Removing a factor of 2^{2003} from numerator and denominator, we have $\dfrac{2^{2004} - 2^{2003}}{2^{2004} + 2^{2003}} = \dfrac{2^{2003}(2-1)}{2^{2003}(2+1)} = \dfrac{1}{3}$.

answer is (c).

7. Let O be the center of the circle. Let A denote the point $(3,0)$ and let B denote the point $(9,0)$. Note that the x-coordinate of the center O is $x = 6$, so that the circle has radius 6. Hence, $OA = OB = 6$ and the triangle OAB is equilateral. Therefore, $\angle AOB = 60^\circ \left(\frac{\pi}{3}\right)$.



So,

$$\begin{aligned}\text{the required area} &= \text{Area of sector } OAB - \text{Area of triangle } OAB \\ &= \frac{1}{2}(6)^2\left(\frac{\pi}{3}\right) - \frac{1}{2}(6)(3\sqrt{3}) \\ &= 6\pi - 9\sqrt{3}\end{aligned}$$

answer is (a).

8. With the given information we have

$$\begin{aligned} a^2b + ab^2 + a + b &= 42 \Rightarrow ab(a+b) + (a+b) = 42 \\ (a+b)(ab+1) &= 42 \Rightarrow 6(a+b) = 42 \;\; \text{since } ab = 5 \\ \Rightarrow a + b &= 7 \end{aligned}$$

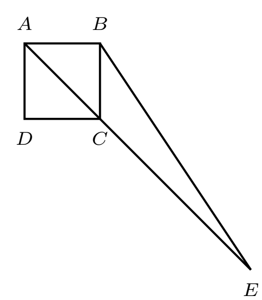
Then $a^2 + b^2 = (a+b)^2 - 2ab = 7^2 - 2(5) = 49 - 10 = 39.$

answer is (d).

9. We consider the three circular sectors swept out upon wrapping the string counterclockwise. Recall the formula $A = \dfrac{\pi\left(wr^2\right)}{180}$ for the area of a circular segment with central angle w (in degrees) and radius r. Since triangle ABC is isosceles, we have $AB = AC = 4$. Then, since angle $ACD = 135$, the area of the first sector is $A_1 = \dfrac{135\pi 10^2}{360} = \dfrac{75\pi}{2}$. Similarly $A_2 = 9\pi$ and $A_3 = \dfrac{\pi}{2}$, giving a total of 48π.

answer is (d).

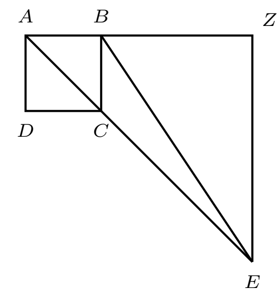
10. Since the square $ABCD$ has area 2, each side of $ABCD$ has length $\sqrt{2}$ and each diagonal of $ABCD$ has length 2. Since AC has length 2, CE has length 4. Also, it is clear that $\angle BCE = 135^\circ$. Then, applying the Law of Cosines to triangle BCE, we have



$$\begin{aligned}\overline{BE}^2 &= \overline{BC}^2 + \overline{CE}^2 - 2\overline{BC} \times \overline{CE}\cos(\angle BCE) \\ &= \left(\sqrt{2}\right)^2 + 4^2 - 2(\sqrt{2})(4)\cos(135^\circ) == 2 + 16 - 8\sqrt{2}(\frac{-1}{\sqrt{2}}) = 2 + 18 + 8 = 26\end{aligned}$$

so that $\overline{BE} = \sqrt{26}$.

Alternate solution: Extend line segment AB. Construct a line through E parallel to segment BC. Suppose these two lines intersect at Z. Then $\angle BZE$ is a right angle and triangle ABC is similar to triangle AZE. Therefore,



$$\frac{\overline{AB}}{\overline{BZ}} = \frac{\overline{AC}}{\overline{CE}} \Rightarrow \frac{\sqrt{2}}{\overline{BZ}} = \frac{2}{4},$$

so $\overline{BZ} = 2\sqrt{2}$. Also,

$$\frac{\overline{BC}}{\overline{ZE}} = \frac{\overline{AC}}{\overline{AE}} \Rightarrow \frac{\sqrt{2}}{\overline{ZE}} = \frac{2}{6},$$

so $\overline{ZE} = 3\sqrt{2}$. Then, by Pythagoras Theorem,

$$\overline{BE}^2 = \overline{BZ}^2 + \overline{ZE}^2 = \left(2\sqrt{2}\right)^2 + (3\sqrt{2})^2 = 8 + 18 = 26,$$

so $\overline{BE} = \sqrt{26}$.

answer is (c).

Junior Final Part B

1. a. We get the integers in alternate rows by adding 8 to the previous entries, so the remainder when we divide by 8 will indicate the column that integer is in. We have $2004 = 250 \times 8 + 4$ so 2004 is in the same column as 4 which is the fifth column.

   $\boxed{\text{Answer: 2004 will fall in the fourth column.}}$

   b. We have $1999 = 249 \times 8 + 7$ so 1999 is in the same column as 7 which is the third column.

   $\boxed{\text{Answer: 1999 will fall in the third column.}}$

   c. Note that
   $$(n+4)^2 = n^2 + 8n + 16$$
   so that $n + 4$ and n have the same remainder when divided by 8. Thus, the remainders when n^2 is divided by 8 repeats in groups of 4. For the first four positive integers the remainders on division by 8 are 1, 4, 1, and 0. Thus, the remainders when $n^2 + 1$ is divided by 8 are 2, 5, 2, and 1. So, numbers of the form $n^2 + 1$ can only fall in the first, second, or fifth columns.

   $\boxed{\text{Answer: } n^2+1 \text{ can only fall in the first, second, or fifth column.}}$

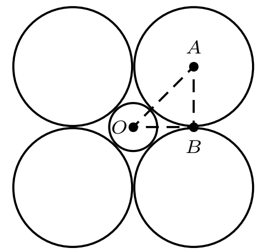
2. If we have the sum of an odd number of consecutive integers, such as $34 + 35 + 36 = 105$, we see that the middle integer is the average of the integers, $\frac{34+35+36}{3} = \frac{105}{3} = 35$. This means that the sum (105) divided by the odd number (of consecutive integers) must be an integer. We have $105 = 3 \times 5 \times 7$ so the odd divisors of 105 are 1, 3, 5, 7, 15, 21, 35, and 105. We eliminate 1 because we need at least two consecutive integers. On the other hand, $\frac{105}{15} = 7$ so 105 is the sum of the 15 consecutive integers $105 = 0 + 1 + \cdots + 13 + 14$ not all of which are positive. In this way we eliminate 15, 21, 35, and 105 consecutive integers leaving 3, 5, and 7.

   If we have the sum of an even number of consecutive integers such as $52 + 53 = 105$, we see that the average is midway between the middle two terms. In other words the average of the integers must be an integer + 0.5, or 105 divided by half the number of terms must be an integer. We have the same divisors as before, so we could have 2, 6, 10, 14, 30, 42, 70 or 210 consecutive integers adding up to 105. The requirement the all the integers be positive leads us to eliminate anything more than 14. There are seven ways to write 105 as a sum of at least two consecutive, positive integers:

number	average	sum
2	52.5	$52+53$
3	35	$34+35+36$
5	21	$19+20+21+22+23$
6	17.5	$15+16+17+18+19+20$
7	15	$12+13+14+15+16+17+18$
10	10.5	$6+7+\cdots+10+11+\cdots+14+15$
14	7.5	$1+2+\cdots+7+8+\cdots+13+14$

Answer: 105 can be written as a sum of at least two consecutive positive integers in 7 ways.

3. The radius of each circle is $OB = AB = 12$, see the diagram. We use Pythagoras to get $OB = 12\sqrt{2}$. If we subtract the radius of the larger circle from this we find that the radius of the smaller circle is $12\sqrt{2} - 12 = 12(\sqrt{2} - 1)$.



Answer: The radius of the smaller circle is $12(\sqrt{2} - 1)$

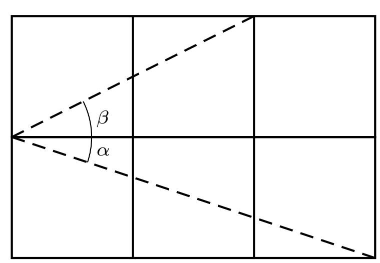
4. a. Every third number in the sequence is even. This is because the first two numbers are odd and the sum of two odds is even so the third entry is even. An odd plus an even is odd making the fourth entry odd. An even plus an odd is odd making the fifth entry odd. We then have two consecutive odd numbers and the pattern repeats. From this we see that f_{51}, f_{150},and f_{300} are even because 51, 150, and 300 are divisible by 3.

Answer: f_{51}, f_{150},and f_{300} are even.

b. We see that every fourth number in the sequence is divisible by three. If we cast out threes (continually subtract 3 from any number greater than 3) from the series we get 1, 1, 2, 0, 2, 2, 1, 0, 1, 1, ... We get the next number in the series by adding the two previous terms and casting out threes (if necessary.) As soon as there is a pair of 1's the series repeats, which happens starting at the ninth term. Every fourth term of the new series is a 0 which indicates that the term in the original series is divisible by 3. Thus f_{48}, f_{196},and f_{1000} are divisible by 3 because 48, 196, and 1000 are divisible by 4.

Answer: f_{48}, f_{196},and f_{1000} are divisible by 3.

5. If we flip the squares as shown in the diagram, we get a triangle with $\alpha + \beta = \angle AOB$. We can use Pythagoras to calculate the lengths of the sides of the triangles: $OA = AB = \sqrt{5}$ and $OB = \sqrt{10}$. From this we see that $OB^2 = OA^2 + AB^2$ so the triangle is an isosceles right triangle with $\angle OAB = 90°$ and $\angle AOB = 45°$.



Answer: $\alpha + \beta = 45°$

Senior Final Part A

1. Numbers normally do not start with the digit '0' so there are nine choices for the first digit. There are no restrictions on the second digit; there are 10 choices for that one. The last two digits are determined by the first two. To get the number of palindromes we multiply the number of choices, $9 \times 10 = 90$.

answer is (b).

2. In any triangle the shortest side is opposite the smallest angle (a consequence of Euclid Book I Proposition 18). In each of the four triangles we are given two of the angles so we have to determine the third by subtracting the given angles from 180°. In $\triangle AOB$ the smallest angle is $\angle OAB = 25°$ so the shortest side is OB. The other three triangles are isosceles (the base angles are equal), so $OB = OC = OD = DE$. Looking at the apex angles we see that only in $\triangle COD$ is there an apex angle less than 60° so the shortest side is CD.

answer is (d).

3. If we make a table of outcomes, listing the possible numbers on the first slip on the left, the possible numbers on the second slip on the top, and the units digit of the sum in the table we obtain

	1	2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8	9	0
2	3	4	5	6	7	8	9	0	1
3	4	5	6	7	8	9	0	1	2
4	5	6	7	8	9	0	1	2	3
5	6	7	8	9	0	1	2	3	4
6	7	8	9	0	1	2	3	4	5
7	8	9	0	1	2	3	4	5	6
8	9	0	1	2	3	4	5	6	7
9	0	1	2	3	4	5	6	7	8

where each of the 81 entries is equiprobable. We find that 0 occurs nine times while each of the other digits occurs only eight times. Thus the most probable digit is 0 with probability $P(0) = \frac{1}{9}$ while $P(1) = P(2) = \cdots = P(9) = \frac{8}{81}$.

answer is (a).

4. We are given that $f(3x) = \dfrac{3}{1+x}$ so $f(x) = \dfrac{3}{1+\dfrac{x}{3}}$ and

$$3f(x) = 3\frac{3}{1+\dfrac{x}{3}} = \frac{27}{3+x}$$

answer is (c).

5. Making a table with the first roll of the die at the side and the second roll at the top we get

	1	2	3	4	5	6
1		×	×	×	×	×
2			×	×	×	×
3				×	×	×
4					×	×
5						×
6						

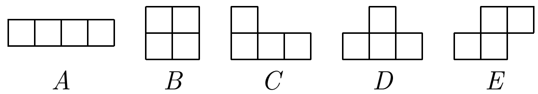
where × indicates that the two outcomes are in strictly increasing order. Each outcome is equiprobable so the required probability can be obtained by dividing the number of ×'s by the number of entries, $\frac{15}{36} = \frac{5}{12}$

answer is (a).

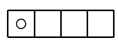
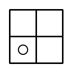
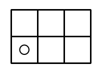
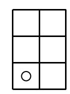
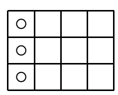
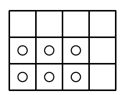
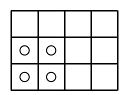
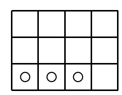
6. If we look at the remainders of the Fibonacci numbers when they are divided by 3 we get 1, 1, 2, 0, 2, 2, 1, 0, 1, 1, 2, 0, ... We get the next number in this sequence by adding the previous two and taking the remainder when divided by 3. We see that the sequence repeats after eight terms. A '0' indicates that the number is divisible by 3; every fourth number in the sequence is a '0' so every fourth Fibonacci number is divisible by 3. In other words, f_4, f_8, f_{12}, ... are divisible by 3. Only 196 is divisible by 4 so only f_{196} is divisible by 3.

answer is (c).

7. There are five possible basic shapes for the four stamps:



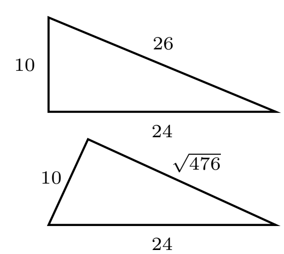
as well as their reflections and rotations. A is a 1×4 rectangle and B is a 2×2 rectangle. If we consider C, D, and E we see that they each fit into a 2×3 rectangle in a number of ways. C fits into such a rectangle four ways (obtained by horizontal and vertical reflections), and D and E each fit into a 2×3 rectangle two ways (obtained by a vertical reflection). Consequently there are eight possible stamp configurations of C, D, and E which fit into a 2×3 rectangle, and, after a 90° rotation, eight possible stamp configurations which fit into a 3×2 rectangle. Now we must count the number of ways each of these rectangles fit into the 3×4 block of stamps. In the following table the $\circ$ in the rectangle can be placed in the block of stamps at the indicated locations:

Rectangle				
Block of stamps				
Number of ways the rectangle fits into the block	3	6	4	3
Number of shapes in the rectangle	1	1	8	8

We obtain our final answer by taking the sum of the number of shapes in each rectangle multiplied by the number of ways the rectangle fits into the 3×4 sheet of stamps. If we do this we get $3 \times 1 + 6 \times 1 + 4 \times 8 + 3 \times 8 = 65$.

answer is (c).

8. We can use Pythagoras to calculate the range of lengths for the third side which give right triangles. The longest is 26 since $10^2 + 24^2 = 100 + 576 = 676 = 26^2$ and the shortest is $\sqrt{476}$ since $24^2 - 10^2 = 576 - 100 = 476$. But $21^2 = 441$ and $22^2 = 484$ so the length of the third side, n, must satisfy $22 \leq n < 26$ so there are four possible values, 22, 23, 24, and 25.



answer is (c).

9. See Junior Final, Part B Problem 2.

answer is (b).

10. See Junior Final, Part B Problem 3.

answer is (d).

Senior Final Part B

1. Suppose that we give the three 3's and two 5's different colours so we can distinguish them. Since a valid number cannot start with a 0, there are $6(6!) = 7! - 6!$ ways to arrange the seven digits into a valid number. On the other hand, in each of these there are 3! ways to arrange the three 3's which are indistinguishable without the colours, and 2! ways to arrange the two 5's which are also equivalent. Thus, the number of arrangements is $\dfrac{6(6!)}{3!2!} = 6 \times 5 \times 4 \times 3 = 360$.

   Answer: There are 360 numbers which can be made using the digits of the number 3053345

2. a. The prime factorization of 2004 is $2004 = 2^2 \times 3^1 \times 167^1$, so the prime factorization of 2004^4 is $2004^4 = 2^8 \times 3^4 \times 167^4$. Thus, any factor of 2004^4 can be written in the form $2^\alpha \times 3^\beta \times 167^\gamma$ where $0 \le \alpha \le 8$, $0 \le \beta \le 4$, and $0 \le \gamma \le 4$. So there are 9 possible values for α and 5 possible values each for β and γ. Hence, by the *multiplication principle of counting*, the total number of possible factors is $9 \times 5 \times 5 = 225$. These include 1 ($\alpha = \beta = \gamma = 0$) and the number 2004^4 ($\alpha = 8, \beta = \gamma = 4$).

   Answer: There are 225 factors of 2004^4.

   b. The product of the 225 factors from part (a) will be of the form $2^\alpha \times 3^\beta \times 167^\gamma$. We need to find α, β, and γ. Each factor will include one of 2^0, 2^1, $\ldots 2^8$. Multiplying these together gives $2^{(0+1+2+\cdots+8)} = 2^{36}$. Each of these possible factors occurs once for each possible combinations of the powers on 3 and 167. Since there are 5 possible powers for both 3 and 167, this gives 25 combinations. So the product of the factors of 2004^4 contains $2^{(36\times25)}$, i.e., $\alpha = 36 \times 25 = 900$. In the same way, multiplying the possible powers of 3 gives $3^{(0+1+\cdots+4)} = 3^{10}$ and, with the 9 possible powers for 2 and 5 for 167, there are 45 possible combinations giving this factor. So, 2004^4 contains $3^{10\times45}$, i.e., $\beta = 10 \times 45 = 450$. The same argument shows that $\gamma = 450$. Thus, the product of all of the factors of 2004^4 equals $2^{900} \times 3^{450} \times 167^{450} = 2004^{450}$.

   Answer: The product of the factors of 2004^4 is 2004^{450}.

3. If light is seen through the two panes of glass, it will either not have been reflected, reflected twice, four times, six times, etc. Since 70% of light is transmitted through each pane and 20% is reflected, we have

$$\begin{aligned} 0.7 \times 0.7 + 0.7 \times 0.2^2 \times 0.7 + 0.7 \times 0.2^4 \times 0.7 + \cdots &= 0.7^2 \times \left(1 + 0.2^2 + 0.2^4 + 0.2^6 + \cdots\right) \\ &= \frac{49}{100} \times \frac{1}{1 - \frac{4}{100}} = \frac{49}{100} \times \frac{100}{100-4} = \frac{49}{96} \end{aligned}$$

   Answer: The total fraction of light that passes through to the other side is $\dfrac{49}{96}$.

4. Given the unusual relation, we can only substitute in astute choices for x and hope for the best. If $x = 2$, $f(2) + 2f\left(\frac{2003}{2-1}\right) = 4013 - 2$, or $f(2) + 2f(2003) = 4011$. If $x = 2003$, $f(2003) + 2f\left(\frac{4004}{2002}\right) = 4013 - 2003$, or $f(2003) + 2f(2) = 2010$. Subtracting twice the first equation from the second, $-3f(2003) = -6012$, or $f(2003) = 2004$.

   Answer: The value of $f(2003)$ is 2004.

5. In the first minute the ant crawls 48 centimetres. The strip then stretches to 2 metres which brings the ant to $48 \times 2 = 96$ centimetres from where it started. In the second minute the ant crawls 48 centimetres to the $96 + 48 = 144$ centimetre mark. The strip then stretches to 3 metres which brings the ant to $144 \times \frac{3}{2} = 216$ centimetres from where it started. In the third minute the ant crawls 48 centimetres to the $216 + 48 = 264$ centimetre mark. The strip then stretches to 4 metres which brings the ant to $264 \times \frac{4}{3} = 352$ centimetres from where it started. In the fourth minute the ant crawls 48 centimetres to the $352 + 48 = 400$ centimetre mark which is the end of the strip. Hence it takes the ant 4 minutes to reach the other end of the strip.

   Answer: It takes 4 minutes to reach the other end