BRITISH COLUMBIA COLLEGES

Junior & Senior High School Mathematics Contest

2005 Solutions

Junior Preliminary

1. Let x_1 and x_2 represent the lengths of the two pieces, and let p_1 and p_2 represent the perimeters of the two squares. Then $\dfrac{p_1}{p_2} = \dfrac{4x_1}{4x_2} = \dfrac{3}{2}$.

Answer is (A).

2. The product is $21 \times 35 \times 15 = 3 \times 7 \times 5 \times 7 \times 3 \times 5 = 3^2 5^2 7^2 = (3 \times 5 \times 7)^2 = 105^2$ which is odd, a multiple of 5, and a perfect square.

Answer is (E).

3. The diameter of such a sphere can be no more than the shortest dimension of the box, and this is 5. Then the radius of the sphere is $\frac{5}{2}$.

Answer is (B).

4. Since the area of a triangle is a linear function of an altitude of the triangle, triangles ABC and ACD have a common altitude, and the associated base of ACD is two-thirds the associated base of ABC, we find the area of ACD is two-thirds the area of ABC; that is 40.

Answer is (C).

5. For $n \neq -5$,

$$\frac{1}{n+5} = 4 \Leftrightarrow n + 5 = \frac{1}{4} \Leftrightarrow n + 6 = \frac{5}{4} \Leftrightarrow \frac{1}{n+6} = \frac{4}{5}$$

Answer is (D).

6. The set of *outcomes* from this *experiment* can be represented by the set of ordered pairs $O = \{(1,1), ..., (6,6)\}$. The subset of O consisting of those ordered pairs whose sum is 5 can be written $A = \{(1,4), (4,1), (2,3), (3,2)\}$. For S a subset of O, we let $n(S)$ denote the number of elements in S and, using the *relative frequency* interpretation of the concept of probability, we define $p(S) = \dfrac{n(S)}{n(O)}$ to be the probability that the "event" S occurs (i.e., the probability that a randomly selected element of O belongs to A). Then we compute $p(A) = \dfrac{n(A)}{n(O)} = \dfrac{4}{36} = \dfrac{1}{9}$.

Answer is (B).

7. The area of the original rectangle is $A_o = 20\,\text{cm} \times 50\,\text{cm} = 1000\,\text{cm}^2$. The new length is $50\,\text{cm} + (50 \times 0.20)\ \text{cm} = 60\,\text{cm}$, and the new width is $20\,\text{cm} - (20 \times 0.20)\ \text{cm} = 16\,\text{cm}$. Then the area of the new rectangle is $A_n = 60\,\text{cm} \times 16\,\text{cm} = 960\,\text{cm}^2$, so $A_n - A_o = 960\,\text{cm}^2 - 1000\,\text{cm}^2 = -40\,\text{cm}^2$. But 40 is 4% of 1000, and then the negative sign then tells us there is a 4% decrease in the area.

Answer is (D).

8. Looking at the five available choices, we see immediately that 435 is divisible by 5 and that 57 = 3 x 17, so neither of these choices is prime. From arithmetic we (may) know that a number is divisible by 3 if and only if the sum of its digits is divisible by 3. Then, since $7 + 8 + 3 = 18$ and $1 + 0 + 9 + 1 + 7 = 18$, the only prime among the choices must be 383.

Answer is (D).

9. Let N denote the number of houses sold in 2003. Then the number sold in 2004 is $N + \dfrac{40N}{100} = \dfrac{7N}{5}$. For this number to be an integer, we must have 5 as a factor of N. The smallest such N is 5, in which case the number of houses sold in 2004 is 7.

Answer is (B).

10. If there are 18 students there could be 9 boys and 9 girls in which case there we cannot ensure at least 10 boys or 10 girls. Suppose there are 19 students. If there are only 9 boys, then there must be 10 girls, and vice-versa. Hence we can ensure at least 10 boys or 10 girls in this case.

Answer is (D).

11. Multiplying the given pair of inequalities by 14, we find $6 < n < \dfrac{28}{3} = 9 + \dfrac{1}{3}$. This pair of inequalities has the three integer solutions 7, 8, and 9.

Answer is (C).

12. The even numbers in this product are 20, 18, $\ldots$, 2. For m an integer write $m = 2^{e_m}n$ where n is odd, and e_m depends on m. Then we want to compute $e_{20!}$. Note that, by the law of exponents, $e_{20!} = e_{20} + e_{18} + e_{16} + ... + e_2$. Also $20 = 2^2 5$, so $e_{20} = 2$, and similarly, $e_{18} = 1$, $e_{16} = 4$, ... , and $e_2 = 1$. Then $e_{20!} = e_{20} + e_{18} + e_{16} + ... + e_2 = 18$.

Answer is (C).

13. Let n, d, and q denote, respectively, the number of nickels, dimes, and quarters, and let x, y, and z denote the total value of each denomination. Then we are given $x + y + z = 2800$, $y - 2z = 0$, and $x - 2y = 0$. Substitution then gives $2y + y + \dfrac{y}{2} = 2800$, so $y = 800, x = 1600$ and $z = 400$. Then $n = \dfrac{800}{10} = 80$, $x = \dfrac{1600}{5} = 320$, and $z = \frac{400}{25} = 16$, giving a total of 416 coins.

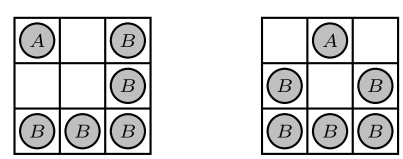
Alternate solution: Let Q be the **value** of the quarters, then the value of the dimes in $2Q$ and, since this is half the value of the nickels, the value of the nickels is $4Q$. Hence $Q + 2Q + 4Q = 2800 \Rightarrow Q = 400$. Thus, there must be \$4 in quarters, \$8 in dimes, and \$16 in nickels. Finally, since there are 4 quarters per dollar, 10 dimes per dollar, and 20 nickels per dollar, the number of coins is $4 \times 4 + 10 \times 8 + 20 \times 16 = 416$.

Answer is (C).

14. Write $a^2 - b^2 = 2005 = 5 \times 401$ for some non-negative integers a and b, and note that 5 and 401 are both prime. Then $(a - b)(a + b) = 5$ x 401, so $a + b = 401$ and $a - b = 5$. This gives $a = 203$ and $b = 198$, and hence $a^2 + b^2 = 203^2 + 198^2 = 41209 + 39204 = 80413$. Note that $198^2 + 203^2$ must end in the same digit as $4 + 9 = 13$, and the only selection that ends in 3 is the correct answer 80413.

Answer is (D).

15. First, note that if the first disk is placed in the center cell, there is no cell in which the second disk can safely be placed. A jump is always possible.
Now, suppose disk A is placed in any one of the other eight cells. In each case, there are precisely 5 positions for disk B in which no jump is possible. The diagrams below show two typical examples, one in which disk A is placed in a "corner" position and one in which disk A is placed in a "non-corner" position.

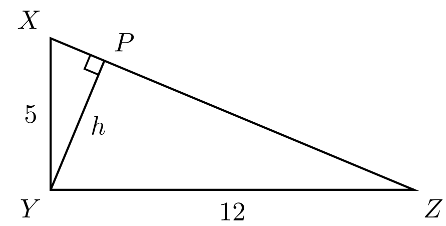


So the total number of "positions" is $8 \times 5 = 40$. But, in making this count, we have distinguished between the position where disk A occupies cell X and disk B occupies cell Y and the position where disks A and B reverse cells. So we have actually counted each position twice. Therefore, the total number of positions where no jump is possible is $\dfrac{1}{2}(40) = 20$.

Answer is (B).

Senior Preliminary

1. By Pythagoras' Theorem, the hypotenuse XZ of triangle XYZ has length $\sqrt{5^2+12^2}=13$. Let YP be a perpendicular from vertex Y to the hypotenuse XZ and suppose that YP has length h. Then, calculating the area A of triangle XYZ in two different ways, we have



$$A=\frac{1}{2}(12)(5)=\frac{1}{2}(13)(h)\Rightarrow 13h=60\Rightarrow h=\frac{60}{13}$$

Answer is (C).

2. Let the larger piece of wire have length x and the smaller piece have length y. We know that $\frac{x}{y}=\frac{3}{2}$.

   The larger square has area $\left(\frac{x}{4}\right)^2=\frac{x^2}{16}$ and the smaller square has area $\left(\frac{y}{4}\right)^2=\frac{y^2}{16}$.

   The ratio of the area of the larger square to the smaller square is $\frac{(x^2/16)}{(y^2/16)}=\frac{x^2}{y^2}=\left(\frac{x}{y}\right)^2=\left(\frac{3}{2}\right)^2=\frac{9}{4}$.

Answer is (B).

3. The number of choices for the leftmost digit is 9 (any of the digits from 1 to 9 inclusive). The number of choices for the next digit is also 9 (any of the digits from 0 to 9 different from the first digit). Similarly, the number of choices for the third digit is 8 and the number of choices for the fourth digit is 7.

   So, the number of four-digit numbers with no repeated digit is $9\cdot 9\cdot 8\cdot 7=4536$.

Answer is (B).

4. If we wish to maximize the sum, we must have the leftmost digits B, D and M either 8's or 9's, so we must have $A=1$. Now, regardless of how we assign the 8 and the two 9's to B, D and M, our three numbers must be of the form 81__ , 91__ and 91__. The sum of these three numbers will be maximized if the units digits D and M are both 9's.

   So we must have $A=1$, $B=8$, $D=9$ and $M=9$. Then the three numbers are $BAD\equiv 819$, $DAM\equiv 919$, and $MAD\equiv 919$, and their sum is 2657.

Answer is (D).

5. If A is a knight, his statement "We are both knaves" is false. But this is a contradiction, since knights always tell the truth. Therefore, A is a knave, and choices (B) and (C) are eliminated.

   Suppose B is also a knave. Then A's statement "We are both knaves" is true. But this is a contradiction, since knaves always lie. Therefore, choice (D) is eliminated.

   So the correct answer must be (A): A is a knave and B is a knight.

Answer is (A).

6. Suppose we have 16 students in a room, 9 boys and 7 girls. The next student who enters the room is either the 10th boy or the 8th girl. So the answer is 17.

Answer is (C).

7. The rectangle with dimensions 20 by 50 has perimeter $2\cdot 20+2\cdot 50=140$.

   If the longer side is increased by 20% and the shorter side is decreased by 20%, the new rectangle has dimensions 16 by 60, and so has perimeter $2\cdot 16+2\cdot 60=152$.

   So, the perimeter has increased by $\frac{152-140}{140}\times 100\%=8\frac{4}{7}\%$.

Answer is (A).

8. Since $6 = 3 \cdot 2$, we need only count the powers of 3 (since there is no shortage of powers of 2). Listing all the integers from 3 to 30 which contain positive powers of 3, we have:

$$\begin{array}{ccccc} 3 = 3^1 & 6 = 3^1 \cdot 2 & 9 = 3^2 & 12 = 3^1 \cdot 4 & 15 = 3^1 \cdot 5 \\ 18 = 3^2 \cdot 2 & 21 = 3^1 \cdot 7 & 24 = 3^1 \cdot 8 & 27 = 3^3 & 30 = 3^1 \cdot 10 \end{array}$$

The highest power of 3 (and hence the highest power of 6) dividing 30! is

$$1 + 1 + 2 + 1 + 1 + 2 + 1 + 1 + 3 + 1 = 14$$

Answer is (C).

9. Since

$$18 < 24 < 36 < 37 < 48$$

taking square roots gives

$$3\sqrt{2} < 2\sqrt{6} < 6 < \sqrt{37} < 4\sqrt{3}$$

and $2\sqrt{6}$ is clearly irrational.

Answer is (E).

10.

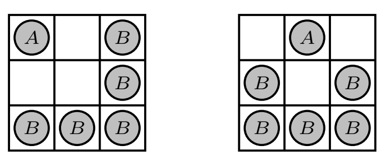
$$\begin{aligned} a^2 - b^2 &= 2005 \\ (a + b) \cdot (a - b) &= 2005 \end{aligned}$$

There are two possibilities: (i) $a+b = 2005$ and $a-b = 1$ or (ii) $a+b = 401$ and $a-b = 5$. Solving the 2 by 2 linear system in case (i) for a and b, we have $a = 1003$, $b = 1002$, which we can eliminate since we know that both a and b are less than 1000. Solving the 2 by 2 linear system in case (ii) for a and b, we have $a = 203$, $b = 198$, which is the solution that we are looking for. Then $a^2+b^2 = 203^2+198^2 = 80413$. Note that $198^2 + 203^2$ must end in the same digit as $4 + 9 = 13$, and the only selection that ends in 3 is the correct answer 80413.

Answer is (D).

11. First, note that if the first disk is placed in the center cell, there is no cell in which the second disk can safely be placed. A jump is always possible.

Now, suppose disk A is placed in any one of the other eight cells. In each case, there are precisely 5 positions for disk B in which no jump is possible. The diagrams below show two typical examples, one in which disk A is placed in a "corner" position and one in which disk A is placed in a "non-corner" position.



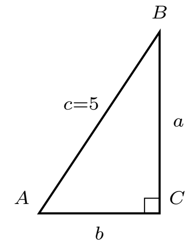
So the total number of "positions" is $8 \times 5 = 40$. But, in making this count, we have distinguished between the position where disk A occupies cell X and disk B occupies cell Y and the position where disks A and B reverse cells. So we have actually counted each position twice. Therefore, the total number of positions where no jump is possible is $\frac{1}{2}(40) = 20$.

The number of ways in which two disks can be placed on the board is $\binom{9}{2} = 36$.

So the probability that no jump is possible is $\frac{20}{36} = \frac{5}{9}$.

Answer is (C).

12. In the diagram the area of triangle ABC is $\frac{1}{2}ab$, so we need calculate ab.



$$c^2 = a^2 + b^2 = (a+b)^2 - 2ab \Rightarrow 5^2 = \left(\sqrt{45}\right)^2 - 2ab$$

so that

$$25 = 45 - 2ab \Rightarrow 2ab = 20 \Rightarrow ab = 10$$

So the area of triangle ABC is $\frac{1}{2}ab = \frac{1}{2}(10) = 5$.

Answer is (D).

13. The area of the circular sector subtended by the angle θ in a circle of radius r is

$$A = \tfrac{1}{2}r^2\theta$$

so the area of any one of the circular sectors ABC, with centers A, B and C, is

$$\text{Area of circular sector } ABC = \frac{1}{2}(3)^2\left(\frac{\pi}{3}\right) = \frac{3\pi}{2}$$

If we multiply this area by 3 the small outer regions are each counted once, but the triangle ABC is counted three times. The area of the triangle is

$$\text{Area of triangle } ABC = \frac{1}{2}(3)\left(\frac{3}{2}\right)\sqrt{3} = \frac{9\sqrt{3}}{4}$$

So the total shaded area is

$$3\left(\frac{3\pi}{2}\right) - 2\left(\frac{9\sqrt{3}}{4}\right) = \tfrac{9}{2}\left(\pi - \sqrt{3}\right)$$

Alternate solution: The area bounded by line segment BC and arc $\widehat{BC}$ is equal to

$$\text{Area of circular sector } ABC - \text{Area of triangle } ABC = \frac{3\pi}{2} - \frac{9\sqrt{3}}{4}$$

Now, there are three such regions, so the area of the three small outer regions is

$$3\left(\frac{3}{4}\left(2\pi - 3\sqrt{3}\right)\right) = \frac{9\pi}{2} - \frac{27\sqrt{3}}{4}$$

Then adding the area of the triangle ABC gives the total shaded area as

$$\begin{aligned}\frac{9\pi}{2} - \frac{27\sqrt{3}}{4} + \frac{9\sqrt{3}}{4} &= \frac{9\pi}{2} - \frac{18\sqrt{3}}{4} \\ &= \tfrac{9}{2}\left(\pi - \sqrt{3}\right)\end{aligned}$$

Answer is (E).

14. Let w_n be the number of walls in n steps.

In going from step $n-1$ to step n , $n \geq 2$, we add 2 "squares" which give 3 new walls each and we add $n-2$ "squares" which give 2 new walls each, for a total of $2 \times 3 + (n-2) \times 2 = 2n + 2$ walls added to w_{n-1} to give w_n .

If the k^{th} step has 340 walls, then

$$\sum_{n=1}^{k} (2n+2) = 2\left(\sum_{n=1}^{k} n\right) + 2k = 2\left(\frac{k(k+1)}{2}\right) + 2k = 340$$

So we have

$$k(k+1) + 2k = 340 \Rightarrow k^2 + 3k - 340 = 0$$

Alternatively, we may use the fact that we add $2n+2$ walls at each stage to construct a table of values of w_n versus n for small values of n

n	w_n	w_n/n
1	4	4
2	10	5
3	18	6
4	28	7
5	40	8
6	54	9
7	70	10

and we may "guess" from this table that, for $n \geq 1$, we have $w_n = n(n+3)$, in which case we are led to the same quadratic equation as before:

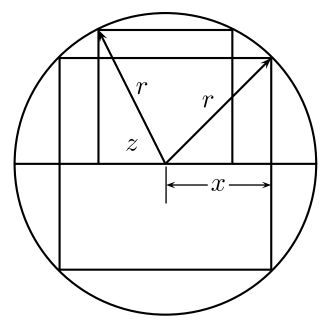
$$n(n+3) = 340 \rightarrow n^2 + 3n - 340 = 0$$

Solving this quadratic equation,

$$n^2 + 3n - 340 = 0 \Rightarrow (n+20)(n-17) = 0 \Rightarrow n = 17$$

That is, a set of 17 steps has 340 walls. Proceeding to a set of 18 steps, we must add $2n+2 = 2(18)+2 = 38$ walls.

Answer is (B).



15. Let x be the half-width of the square inscribed in the the circle (the larger square), and let z be the half-width of the square inscribed in the the semicircle (the smaller square). From the diagram, we have $r^2 = x^2 + x^2 = 2x^2$ and $r^2 = z^2 + (2z)^2 = 5z^2$. Therefore, $2x^2 = 5z^2$, so that $4x^2 = 10z^2 = \frac{5}{2}\left(4z^2\right)$. That is,

$$\text{Area of larger square} = \frac{5}{2}\,(\text{Area of smaller square})$$

So, if the area of the larger square is $15\,\text{cm}^2$, the area of the smaller square is $6\,\text{cm}^2$.

Answer is (C).

Junior Final, Part A

1. $2 \diamond (3 \star 3) = 2 \diamond 1 = 3$

Answer is (B).

2. The first person might be given any of the 3 coats leaving two for the second and one for the third, so there are $3 \times 2 \times 1 = 6$ ways of distributing the coats. We can list the six different ways to distribute the coats (we use $(2, 1, 3)$ to indicate that the first person gets the second person's coat, the second person gets the first person's coat, and the third person gets his own coat): $(1, 2, 3)$, $(1, 3, 2)$, $(2, 1, 3)$, $(2, 3, 1)$, $(3, 1, 2)$, $(3, 2, 1)$. Only two of these have everybody receiving someone else's coat: $(2, 3, 1)$ and $(3, 1, 2)$. The the probability that no one gets the correct coat is $\frac{2}{6} = \frac{1}{3}$.

Answer is (B).

3. If we count a weight as being positive if it is on the side opposite the object being weighed and negative if is on the same side, we see that:

(A) 1, 5, 7: We cannot obtain 9 or 10.

(B) 2, 5, 6: We cannot obtain 10 or 12.

(C) 1, 3, 9: $1 = 1$, $2 = 3 - 1$, $3 = 3$, $4 = 3 + 1$, $5 = 9 - 3 - 1$, $6 = 9 - 3$, $7 = 9 + 1 - 3$, $8 = 9 - 1$, $9 = 9$, $10 = 9 + 1$, $11 = 9 + 3 - 1$, $12 = 9 + 3$, $13 = 1 + 3 + 9$.

(D) 2, 4, 7: We cannot obtain 8, 10 or 12.

(E) 3, 4, 6: We cannot obtain 8, 11 or 12.

Answer is (C).

4. We need to draw 50 cards to ensure that at least two are aces: there are 48 non-aces which we could draw before the two aces. We need 9 to ensure that three are the same suit: we could draw two of each of the four suits before we get the third card of some suit. Thus drawing 9 cards will ensure that two are aces or three are the same suit (in fact, three will be the same suit.)

Answer is (A).

5. The four discs in the corners cannot be removed, leaving 5 in the shape of a cross which can. Consequently, the maximum possible number which might result in a winnable game is six (the five which can be removed and one left over.) If we start with 5 in the shape of a cross and add a sixth is to a corner, it can jump to the other corners along the edges and then over the central one, showing that six can be placed on the board to give a winning game.

Answer is (D).

6. Since the factors in the product come in pairs alternating between addition and subtraction, and the first and last terms have an addition, there is a odd number of factors. Further, each pair of the factors we have

$$\left(1 + \frac{1}{i}\right)\left(1 - \frac{1}{i+1}\right) = 1 + \frac{1}{i} - \frac{1}{i+1} - \frac{1}{i(i+1)} = 1 + \frac{i + 1 - i - 1}{i(i+1)} = 1$$

Thus, each pair of factors multiplies out to a value of 1, so the only factor that remains is the last. So the product equals

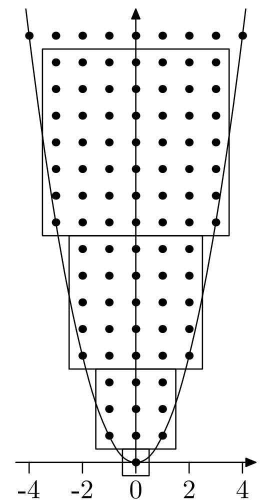
$$1 + \frac{1}{n} = \frac{n+1}{n}$$

Answer is (D).

7. An integer between 500 and 600 is of the form $5ab$ where a and b are integers between 0 and 9. For the sum of the digits to be 12, it must be true that $a + b = 7$. The solutions (a, b) to the equation $a + b = 7$ for which $0 \le a, b \le 7$ are: $(0, 7)$, $(1, 6)$, $(2, 5)$, $(3, 4)$, $(4, 3)$, $(5, 2)$, $(6, 1)$, and $(7, 0)$. This gives a total of eight possible numbers.

Answer is (C).

8. We see from the diagram that the grid points lie in a series of squares. The point with $y=0$ lies in a 1×1 square, the points with $1\le y\le 3$ lie in a 3×3 square, the points with $4\le y\le 8$ lie in a 5×5 square, and so on up to the points with $36\le y\le 48$ which lie in a 13×13 square. This leaves the grid points with $y=49$ or $y=50$ which lie in a 2×15 rectangle. The total number of grid points is therefore $1^2+3^2+5^2+\cdots+13^2+2\cdot 15=485$.

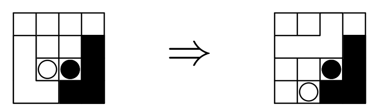


Alternate solution: Looking at the vertical lines of grid points we have: a total of 51 grid points on the y-axis, $x=0$, from $y=0$ to $y=50$; $51-1=50$ grid points along the lines $x=1$ and $x=-1$ from $y=1$ to $y=50$; $51-4=47$ grid points along the lines $x=2$ and $x=-2$ from $y=4$ to $y=50$; and so on to $51-49=2$ grid points along the lines $x=7$ and $x=-7$ from $y=49$ to $y=50$. This gives a total of

$$\begin{aligned}51+2\,(50+47+42\,+\,&35+26+15+2)\\ &=51+2\,[7\cdot 51-(1+4+16+25+36+49)]\\ &=51+2\left[7\cdot 51-\frac{(7)(7+1)(2\cdot 7+1)}{6}\right]=485\end{aligned}$$

Answer is (B).

9. Consider the diagram. For the position on the right, black cannot place the L-shaped piece in any other position than its current position without covering a square that is already occupied by a disc or the other L-shaped piece.



Answer is (B).

10. The simple plan suggested in the problem of starting at the first floor and going up one floor at a time does not make use of the fact that you have two eggs, but it is the only plan that can be used if you have only one egg.

Start at the second floor and go up two floors at a time. If the first egg breaks at floor two, the critical height is one or two. Then, dropping the second egg from floor one determines which it is, for a total of two drops. If the first egg does not break at floor two but does break at floor four, the critical height is three or four. Then, dropping the second egg at floor three determines the critical height, for a total of three drops. As you go up every second floor, the maximum number of drops increases by one. So if the first egg breaks at floor 36 the the first egg breaks, the maximum number of drops is 19.

Start at the third floor and go up three floors at a time. If the first egg breaks at floor three, the critical height is one, two, or three. At most two more drops (second egg from floors one and two) determines it, for a total of at most 3 drops. If the first egg does not break at floor three but does break at floor 6, the critical height is 4, 5, or 6. At most two more drops are required, for a at most 4 drops. Again, as you go up every third floor, the maximum number of drops goes up by one. So if first egg breaks at floor 36, the maximum number of drops is 14.

In the same way starting at the fourth floor and going up four floors at a time, gives a maximum of 12 drop; starting at the fifth, sixth, seventh, or eighth floors and going up five, six, seven, or eight floors at a time, respectively, gives a maximum of 11 drops; starting at the ninth floor and going up nine floors at a time gives a maximum of 12 drops.

The plans described above show that if you go up the same number of floors every time, your maximum number of drops increases by one every time you go up another step. So instead, find a number of floors, n, for which you start at floor n, and if the egg does not break at that floor, go up one less floor at the next step. This, keeps the maximum number of drops at n for every step. So we need to find n for which

$$n+(n-1)+(n-2)+\cdots+1=36$$

Problem 10 continued ...

...Problem 10 continued

since if we reach floor 36 with breaking the first egg we know the critical height is 37. This gives

$$\frac{n(n+1)}{2} = 36 \Rightarrow n^2 + n - 72 = 0 \Rightarrow (n-8)(n+9) = 0$$

The only valid solution is $n = 8$. Thus, start at floor 8. If the first egg breaks, at most 8 drops are required. If it does not break, go up 7 more floors to floor 15. If the first egg breaks, at most 8 drops are required, and so on. This plans gives a maximum of 8 drops.

Alternate solution: Here is a more formal way to derive the answer above. Suppose that we know that the critical height is between floors 1 and n inclusive and $M(n)$ is the minimum number of times you must drop the eggs in order to guarantee the successful determination of the critical height. Then $M(1) = 0$ (we know the critical height is between 1 and 1) and we can define $M(n)$ recursively as follows: suppose that we drop an egg from height i and it breaks. Then we can start dropping the other egg at floor 1 and proceed upwards until it breaks, for a maximum of i trials. If it does not break then we replace the given problem with one involving $n-i$ floors for which the answer is $M(n-i)$. The most we might need is $\max(i, 1 + M(n-i))$. We then get the value for $M(n) = \min_{1\le i<n}(\max(i, 1 + M(n-i)))$. We can also read the procedure used to realize this minimum from the choice(s) which gives the minimum. We will represent a procedure by an ascending series of numbers which should be interpreted as follows: drop the egg from the floors indicated by the series until it breaks. Then continue to drop it from the floor which is one above the floor where the egg did not break most recently. If the egg has not broken by the end of the series then the critical height is the next floor. See the table below for the calculations. The minimum number of drops is 8 realized by the (unique) procedure 8, 15, 21, 26, 30, 33, 35, 36.

n	Values giving minimum	$M(n)$	Procedure(s)
1		0	
2	$\max(1, 1 + M(0) = 1)$	1	1
3	$\max(1, 1 + M(2) = 2) = 2$ $\max(2, 1 + M(1) = 1) = 2$	2	1,2; 2
4	$\max(2, 1 + M(2) = 2)$	2	2,3
5–6	...	3	...
7	$\max(3, 1 + M(4) = 3)$	3	3,5,6
8–10	...	4	...
11	$\max(4, 1 + M(7) = 4)$	4	4,7,9,10
12–15	...	5	...
16	$\max(5, 1 + M(11) = 5)$	5	5,9,12,14,15
17–21	...	6	...
22	$\max(6, 1 + M(16) = 6)$	6	6,11,15,18,20,21
23–28	...	7	...
29	$\max(7, 1 + M(22) = 7)$	7	7,13,18,22,25,27,28
30–36	...	8	...
37	$\max(8, 1 + M(29) = 8)$	8	8,15,21,26,30,33,35,36

Answer is (A).

Junior Final, Part B

1. Since $\triangle ABF$ is a right triangle and we know that $\overline{AB} = 17$ and $\overline{BF} = 8$, we can use Pythagoras to find $\overline{AF} = \sqrt{17^2 - 8^2} = \sqrt{(17-8)(17+8)} = \sqrt{9 \times 25} = 15$. Then $\overline{EF} = \overline{AF} - \overline{AE} = 7$ so the area of the shaded square is $7^2 = 49$.

   Alternate solution: Since $\overline{FB} = 8$ and $\overline{AF} = 15$, the area of each of the four congruent triangles is $\frac{1}{2}(8)(15) = 60$. Hence, the area of the shaded square is $17^2 - 4(60) = 289 - 240 = 49$.

   Answer: The area of the shaded quadrilateral $EFGH$ is 49.

2. Let n be the number in the party and b be the bill. Then we have a system of two equations in two unknowns which we can solve in a variety of ways.

$$\begin{aligned} 16n &= b - 4 \\ 19n &= 1.15b + 2 \end{aligned} \Rightarrow \begin{aligned} 16n &= b - 4 \\ 3n &= 0.15b + 6 \end{aligned} \Rightarrow \begin{aligned} 16n &= b - 4 \\ n &= 0.05b + 2 \end{aligned}$$

   so that

$$\begin{aligned} 0.8b + 32 &= b - 4 \\ n &= 0.05b + 2 \end{aligned} \Rightarrow \begin{aligned} 0.2b &= 36 \\ n &= 0.05b + 2 \end{aligned} \Rightarrow \begin{aligned} b &= 5(36) = 180 \\ n &= 0.05(180) + 2 = 11 \end{aligned}$$

   Alternate solution: Starting from $16n = b - 4 \Rightarrow b = 16n + 4$ and $19n = 1.15b + 2$ gives

$$19n = 1.15(16n + 4) + 2 = 18.4n + 6.6 \Rightarrow 0.6n = 6.6 \Rightarrow n = 11$$

   Then, $b = 16(11) + 4 = 176 + 4 = 180$.

   Answer: There were 11 in the party and the bill was $180.

3. First, since $x^2 > 0$, $y^3 > 0$, so $y > 0$. We are given that $x^2y^3 = 6^{12} = 2^{12}3^{12}$. Since x and y both divide $2^{12}3^{12}$, we must have $x = \pm 2^i 3^j$ and $y = 2^k 3^l$, where i, j, k, l are integers and $0 \le i, j, k, l \le 12$.

   Since $x = \pm 2^i 3^j$, $x^2 = 2^{2i}3^{2j}$, and, since $y = 2^k 3^l$, $y^3 = 2^{3k}3^{3l}$. Therefore, $x^2y^3 = 2^{2i+3k}3^{2j+3l}$, so that $2i + 3k = 12$ and $2j + 3l = 12$. Solving these equations, we find that $(i, k) = (0, 4)$, $(3, 2)$, or $(6, 0)$, and that $(j, l) = (0, 4)$, $(3, 2)$, or $(6, 0)$.

   Now, each of the 3 values of i can be paired with either of the 3 values of j. Once this is done, the values of k and l are determined. So the number of solutions in positive integers is $3 \times 3 = 9$ and the total number of solutions is $9 \times 2 = 18$.

   Note that it is now an easy matter to list all the solutions. They are

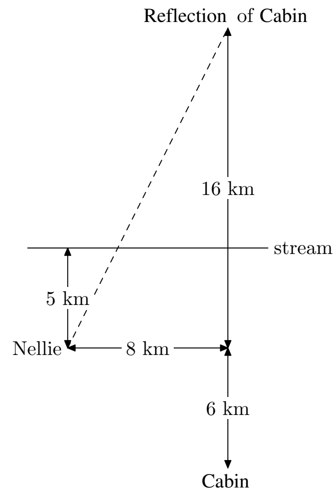
$$\begin{aligned} (x, y) = &(1, 1296), (-1, 1296), (8, 324), (-8, 324), (27, 144), (-27, 144), (64, 81), \\ &(-64, 81), (216, 36), (-216, 36), (729, 16), (-729, 16), (1728, 9), (-1728, 9), \\ &(5832, 4), (-5832, 4), (46656, 1), (-46656, 1) \end{aligned}$$

   This is not a required part of the solution.

   Answer: There are 18 integer solutions to the equation $x^2y^3 = 6^{12}$.

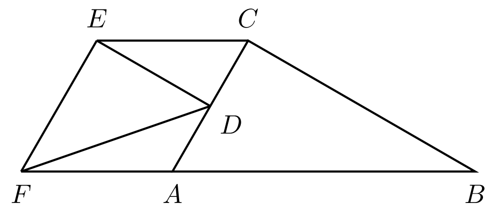
4. Nellie's shortest route will be a straight line to the stream followed by a straight line to her cabin. If we reflect the second part of her trip to the cabin in the stream, it is clear that the shortest distance will be the straight line from where Nellie starts out to the reflection of the cabin. We use Pythagoras to calculate this distance as

$$\sqrt{8^2 + 16^2} = 8\sqrt{5}\,\text{km}$$



Answer: The shortest distance that Nellie must travel is $8\sqrt{5}$ km.

5. In the diagram triangle ABC is a $30°$-$60°$-$90°$ triangle with the right angle at vertex C, the $30°$ angle at vertex B, and side AB having length 20. Segment ED is perpendicular to side AC and D bisects AC. Segment EC is parallel to AB. Segment EF is perpendicular to ED and F is on the extension of AB.



(a) In triangle ABC side AB is the hypotenuse so that $\overline{AC} = 10$, and, since D bisects AC we have $\overline{CD} = 5$. Since ED is perpendicular to AC and EC is parallel to AB, triangle CDE is $30°$-$60°$-$90°$ with the $30°$ angle at E and hypotenuse EC. Thus, $\overline{EC} = 10$ and by Pythagoras' Theorem

$$\overline{ED}^2 = 10^2 - 5^2 = 75 \Rightarrow \overline{ED} = 5\sqrt{3}$$

Answer: $\overline{ED} = 5\sqrt{3}$

(b) Since EF is perpendicular to ED, it is parallel to AC. Further, AF is parallel to EC so that $ACEF$ is a rhombus with all sides equal to 10. Applying Pythagoras's Theorem to right triangle DEF gives

$$\overline{DF}^2 = \overline{EF}^2 + \overline{DE}^2 = 10^2 + \left(5\sqrt{3}\right)^2 = 175 \Rightarrow \overline{DF} = \sqrt{175} = 5\sqrt{7}$$

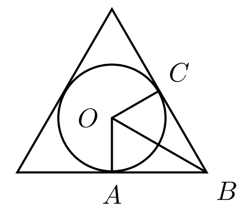
Answer: $\overline{ED} = 5\sqrt{7}$

Senior Final, Part A

1. An integer between 500 and 600 is of the form $5ab$ where a and b are integers between 0 and 9. For the sum of the digits to be 12, it must be true that $a + b = 7$. The solutions (a, b) to the equation $a + b = 7$ for which $0 \le a, b \le 7$ are: $(0, 7)$, $(1, 6)$, $(2, 5)$, $(3, 4)$, $(4, 3)$, $(5, 2)$, $(6, 1)$, and $(7, 0)$. This gives a total of eight possible numbers.

Answer is (C).

2. The inscribed sphere will touch the side of the cone along a circle and will be tangent to the base of the cone. If we cut the cone and inscribed sphere with a plane along the axis of the cone, an isosceles triangle with a circle inscribed will result. In fact, since the base of the triangle is the diameter of the base of the cone, which $2 \times 5\,\text{cm} = 10\,\text{cm}$, the triangle is equilateral. See the diagram. Since lines AB and BC are tangent to the circle, the line BO bisects the angle $\angle ABC = 60°$. Thus, ABO is a 30°-60°-90° triangle with the 30° angle at B.

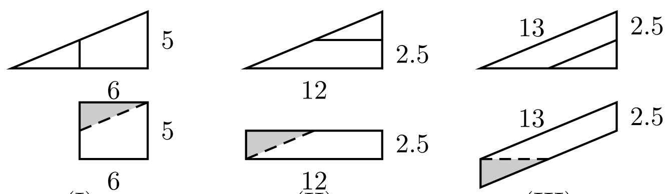


Further, $\overline{AB} = 5$ and OA is the radius r of the inscribed circle, and so the radius of the sphere inscribed in the cone. The standard 30°-60°-90° has sides 2, 1, and $\sqrt{3}$. So by similar triangles

$$\frac{\overline{OA}}{\overline{AB}} = \frac{r}{5} = \frac{1}{\sqrt{3}} \Rightarrow r = \frac{5}{\sqrt{3}} = \frac{5\sqrt{3}}{3}$$

Answer is (B).

3. Note that since $13^2 = 12^2 + 5^2$ we are dealing with a right triangle. There are three possible ways to cut the triangle into two pieces that can be formed into a quadrilateral, in each case the cut is parallel to one side and bisects the other two sides. The possibilities are shown below:



(I) (II) (III)

Possibility (III) is not a rectangle. The perimeters in (I) and (II) are 22 and 29, respectively. Thus, the maximum possible perimeter is 29.

Answer is (B).

4. From the given information we have

$$\begin{aligned}
4 + A + B = 19 &\quad \Rightarrow A + B = 15 \\
A + B + C = 19 &\quad \Rightarrow C = 4 \\
B + C + D = 19 &\quad \Rightarrow B + D = 15 \\
C + D + E = 19 &\quad \Rightarrow D + E = 15 \\
D + E + F = 19 &\quad \Rightarrow F = 4 \\
E + F + G = 19 &\quad \Rightarrow E + G = 15 \\
F + G + 8 = 19 &\quad \Rightarrow F + G = 11
\end{aligned}$$

Now $F = 4$ and $F + G = 11$ give $G = 4$. Then, $E + G = 15$ gives $E = 8$. Finally, $D + E = 15$ gives $D = 7$.

Answer is (C).

5. Since the factors in the product come in pairs alternating between addition and subtraction, and the first and last terms have an addition, there is a odd number of factors. Further, each pair of the factors we have

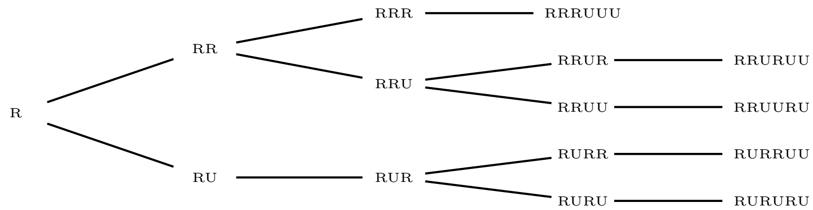
$$\left(1 + \frac{1}{i}\right)\left(1 - \frac{1}{i+1}\right) = 1 + \frac{1}{i} - \frac{1}{i+1} - \frac{1}{i(i+1)} = 1 + \frac{i + 1 - i - 1}{i(i+1)} = 1$$

Thus, each pair of factors multiplies out to a value of 1, so the only factor that remains is the last. So the product equals

$$1 + \frac{1}{n} = \frac{n+1}{n}$$

Answer is (D).

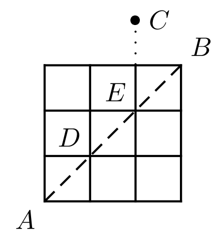
6. Represent the six people in the line by a sequence of three R's, the people with the Right change of \$10, and three U's, the people with the Unchangeable \$20 bills. The first letter in the sequence must be R, otherwise the cashier won't be able to make change for the first person. The second letter can be either R or U, since now the cashier has one \$10 bill. The subsequent possibilities can be modelled using the decision tree below:



Thus, there are five possible sequences RRRUUU, RRURUU, RRUURU, RURRUU, and RURURU. So that there are five possible ways for the six people to line up in such a way that the person selling the ticket always has enough change. Finally, the total number of ways for the six people to line up is the number of ways of selecting three of the six places in the line for the three R's, which equals

$$\binom{6}{3} = \frac{6!}{3!3!} = 20$$

So the probability that the six people will line up in such a way that the person selling the ticket always has enough change is $\dfrac{5}{20} = \dfrac{1}{4}$



Alternate solutions: Another way to model this problem is to consider the 3×3 grid shown. A path from point A to point B that does not allow any backtracking can be represented as a sequence of three R's, go right at an intersection, and three U's, go up at an intersection. If the path does not go above the diagonal line, then, reading from left to right, the sequence can never have more U's than R's to the left of any position in the sequence. This is the same as the requirement for the sequence to represent one in which the cashier can always make change. The number of paths from A to B that do not go above the diagonal line can be determined by first finding the number of paths that do and then subtracting from the total number of paths.

To count the number of paths that do go above the diagonal note that for any path that goes above the diagonal the sequence of R's and U's representing the path must have more U's than R's to the left of some position in the sequence. For example, in the sequence RUUR̲UR there are more U's to the left of the underlined position. Transform the sequence by changing all the R's to U's and all the U's to R's from this position to the right hand end of the sequence. For the example above this gives the sequence RUUURU. This sequence goes from point A to the point C, since there are now two R's and four U's. Any sequence representing a path that goes above the diagonal is transformed in this way into a sequence that represents a path the goes from point A to point C, since the sequence from the point where the number of U's exceeds the number of R's will have exactly one extra R and the transformation gives exactly one extra U from this point and the initial sequence gives one additional extra U. Hence, the number of sequences representing paths from A to B that go above the diagonal is equal to or less than the number of paths that go from A to C.

Further, any path that goes from A to C must go above the diagonal line and so the same transformation can be performed on such a path. For example, the path represented by UR̲URUU goes from A to C and the number of U's exceeds the number of R's from the underlined position on. The transformation gives UURURR which represents a path from A to B, equal number of R's and U's, that goes above the diagonal. Any path from A to C can be transformed in the same way into a path from A to B that goes above the diagonal. Hence, the number of paths from A to C is less than equal to the number of paths from A to B that go above the diagonal. Putting the two results above together we see that the number of paths from A to B that go above the diagonal equals the number of paths from A to C.

Problem 6 continued ...

. . . Problem 6 continued

Finally, the number of paths from A to B equals the number of ways of placing the three R's in the six possible positions in the sequence, which equals

$$\binom{6}{3} = \frac{6!}{3!3!} = 20$$

and number of paths from A to C is the number of ways to place the two R's in the six positions of the sequence (or the number of ways to place the four U's in the six positions), which equals

$$\binom{6}{2} = \binom{6}{4} = \frac{6!}{2!4!} = 15$$

So, the number of paths from A to B that do not go above the diagonal equals

$$\binom{6}{3} - \binom{6}{2} = \frac{6!}{3!3!} - \frac{6!}{2!4!} = \frac{4 \cdot 6!}{4\,(3!3!)} - \frac{3 \cdot 6!}{3\,(2!4!)} = \frac{4 \cdot 6!}{4\,(3!3!)} - \frac{3 \cdot 6!}{4\,(3!3!)} = \frac{6!}{4\,(3!3!))} = \frac{1}{4}\binom{6}{3} = 5$$

In general, if the there are n people with \$10 bills and n people with \$20 bills, the number of possible ways for them to line up so that the cashier can always make change is

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

The number C_n is called a Catlan number.

If you can stand it, here is another solution. Break the set of paths from A to B that do not cross the diagonal into groups based on where the path first hits the diagonal: the points D, E, or B. For the sequences of R's and U's this is the position in the sequence where the number of U's equals the number of R's from that position to the left. Note that this can only take place at an even numbered position. Break the original sequence at this point and remove the left most R and the right most U from the sequence on the left. If the path first hits the diagonal at point D, then this happens at position 2 and the left sequence has no R' or U's and the right has two R's and two U's. This represents a 0×0 grid and a 2×2 grid. If C_0 is the number of paths in a 0×0 grid, take it to be 1, and C_2 is the number of sequences in a 2×2 grid, by the multiplication principle the number of paths from A to B the do not go above the diagonal but hit the diagonal at point D is C_0C_2. In the same way, if the path first hits the diagonal at E the number of paths is C_1C_1, and if the path first the diagonal at B the number of paths is C_2C_0. Since these are disjoint possibilities, the total number of paths from A to B that do not go above the diagonal is

$$C_3 = C_0C_2 + C_1C_1 + C_2C_0$$

By simple counting, or by applying the corresponding formula for C_1 and C_2 (with $C_0 = 1$), we see that $C_1 = 1$ and $C_2 = 2$. Then the formula above gives

$$C_3 = (1)(2) + (1)(1) + (2)(1) = 5$$

In general, for the Catlan number C_n, which represents the number of paths from A to B in an $n \times n$ grid that do not go above the diagonal, satisfies the recursion formula

$$C_n = C_0C_{n-1} + C_1C_{n-2} + C_2C_{n-3} + \cdots + C_iC_{n-i-1} + \cdots + C_{n-1}C_0$$

Answer is (A).

7. Since AFB is a right triangle with $\overline{AB} = 17$ and $\overline{FB} = 8$, we must have

$$\overline{AF}^2 = 17^2 - 8^2 = 188 = 64 = 125 = 15^2 \Rightarrow \overline{AF} = 15$$

Since triangle ADE is congruent to triangle AFB, $\overline{AE} = \overline{FB} = 8$, so that $\overline{EF} = 7$. By a similar argument $\overline{FG} = \overline{GH} = \overline{HE} = 7$. Further, since the triangles are all right triangles, the angles at E, F, G, and H are all right angles. Hence, $EFGH$ is a square with area $7^2 = 49$.

Alternate solution: Since $\overline{FB} = 8$ and $\overline{AF} = 15$, the area of each of the four congruent triangles is $\frac{1}{2}(8)(15) = 60$. Hence, the area of the shaded square is $17^2 - 4(60) = 289 - 240 = 49$.

Answer is (B).

8. Note that a disc at a corner square cannot be jumped, and a disc on any other square can be. Thus, placing five discs on the non-corner squares and one disc on one of the corner squares allows the disc on the corner square to jump all of the other discs, leaving a single un-jumped disc on the corner opposite where the original jumping disc started.

Answer is (D).

9. The n^{th} term in the sequence is $3 + 4(n-1) = 4n - 1$, and

$$4n - 1 = 17655 \Rightarrow 4n = 17656 \Rightarrow n = 4414$$

Thus, 17655 is the 4414^{th} number in the sequence. There are $20 \cdot 30 = 600$ numbers per page and

$$\frac{4414}{600} = 7 + \frac{107}{300}$$

So, 17655 is on page 8.

Answer is (E).

10. There are 16 possible scores, 0 through 15. If 29 students each get one of these 16 possible scores, there would be 464 students and no score would be repeated 30 or more times. If there is one more student, at least one score must be repeated at least 30 times. Thus, in order to guarantee that at least 30 students get the same final score, there must be 465 students.

Answer is (D).

Senior Final, Part B

1. Since the three digit number bcd is divisible by 5 and only the digits 1, 2, 3, 4, and 5 are used, the digit d must be 5. Further, since the three digit number abc is divisible by 4, the digit c must be even. So c can only be 2 or 4. Hence, the three digit number cde is divisible by 3, has a middle digit, d, equal to 5, and a leading digit, c, that is either 2 or 4. The only possibilities, using only the digits 1, 2, 3, 4, and 5, are 252, 255, and 453. Only the last one meets the requirement that the digits are only used once. Thus, $c = 4$, $d = 5$, and $e = 3$. So, a and b can only be 1 or 2. This gives only two possibilities for the three digit number abc, 124 or 214. Only 124 is divisible by 4, so $a = 1$.

$\boxed{\textbf{Answer: } a = 1}$

2. (a) There is a total of 13 balls in the urn and six of them are red. Thus, the probability of selecting a red ball when a single ball is drawn is

$$P\,(\text{one red ball selected when one ball is selected}) = \frac{6}{13}$$

$\boxed{\textbf{Answer: Probability is } \frac{6}{13}\textbf{.}}$

Problem 2 continued ...

...Problem 2 continued

(b) The number of ways of selecting two of the 13 balls is

$$\binom{13}{2} = \frac{13 \cdot 12}{2 \cdot 1} = 13 \cdot 6 = 78$$

The number of ways of selecting two of the black balls is

$$\binom{4}{2} = \frac{4 \cdot 3}{2 \cdot 1} = 6$$

Thus, the probability of getting two black balls when two balls are drawn is

$$P\,(\text{two black balls selected when two balls are selected}) = \frac{6}{78} = \frac{1}{13}$$

Answer: Probability is $\frac{1}{13}$.

(c) If it is known that the two balls drawn are of the same colour, then the sample space consists only of those cases in which both balls are of the same colour. This the number of ways of choosing two white balls, or two red balls, or two black balls. This gives a total of

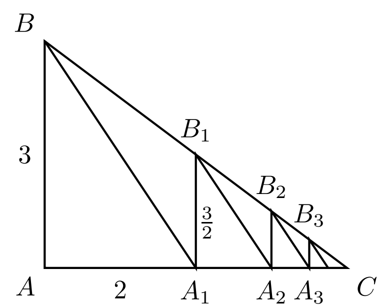
$$\binom{3}{2} + \binom{6}{2} + \binom{4}{2} = 3 + 15 + 6 = 24$$

Thus, the probability of selecting two black balls if it is known that the two balls drawn are of the same colour is

$$P\,(\text{two black balls selected given two balls of the same colour are selected}) = \frac{6}{24} = \frac{1}{4}$$

Answer: Probability is $\frac{1}{4}$.

3. Since A_1 bisects line segment AC, the length of the segment AA_1 is $\overline{AA_1} = 2$ and the length of the segment B_1A_1 is $\overline{B_1A_1} = \frac{3}{2}$. See the diagram. The length of the segment $\overline{BA_1}$ is given by Pythagoras' Theorem as



$$\overline{BA_1} = \sqrt{\overline{BA}^2 + \overline{AA_1}^2} = \sqrt{3^2 + 2^2} = \sqrt{13}$$

From here on note that each of the triangles $B_iA_iA_{i+1}$ is similar to the triangle BAA_1 with all dimensions reduced by a factor of $\frac{1}{2}$ at each step. Thus, the total length of the sequence of diagonal segments is

$$\begin{aligned}
&\sqrt{13} + \frac{\sqrt{13}}{2} + \frac{\sqrt{13}}{2^2} + \cdots + \frac{\sqrt{13}}{2^n} + \cdots \\
&= \sqrt{13}\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^i} + \cdots\right) \\
&= \sqrt{13}\left(\frac{1}{1 - \frac{1}{2}}\right) = 2\sqrt{13}
\end{aligned}$$

Answer: Total length of sequence of diagonal segments is $2\sqrt{13}$

4. Since α and β are the real roots of the equation, it can be written as

$$(x-\alpha)(x-\beta) = x^2 - (\alpha+\beta) + \alpha\beta$$

So $\alpha+\beta = 3$ and $\alpha\beta = q$. Now

$$\alpha^3+\beta^3 = (\alpha+\beta)\left(\alpha^2-\alpha\beta+\beta^2\right) = 3\left(\alpha^2-\alpha\beta+\beta^2\right)$$

so that $\alpha^3+\beta^3 = 81 \Rightarrow \left(\alpha^2-\alpha\beta+\beta^2\right) = \left(\alpha^2+\beta^2-q\right) = 27$. Squaring $\alpha+\beta$ give

$$(\alpha+\beta)^2 = \alpha^2+2\alpha\beta+\beta^2 = \alpha^2+\beta^2+2q = 9$$

Let $Z = \alpha^2+\beta^2$, then we have two equations for q and Z.

$$\begin{aligned} Z - q &= 27 \\ Z + 2q &= 9 \end{aligned}$$

Multiplying the top equation by -1 and adding gives $3q = -18 \Rightarrow q = -6$.

Answer: $q = -6$

5. Let ab and tu be the present ages of Anne and Tom, and let AB and TU be their ages 31 years later. Then,

$$AB - ab = 10A + B - (10a+b) = 10(A-a) + B - b = 31$$

and

$$TU - tu = 10(T-t) + U - u = 31$$

Further, $abtu$ and $ABTU$ are both perfect squares, so that there integers m and n such that

$$\begin{aligned} 1000a + 100b + 10c + d &= m^2 \\ 1000A + 100B + 10C + D &= n^2 \end{aligned}$$

Subtracting the top equation from the bottom gives

$$1000(A-a)+100(B-b)+10(T-t)+U-u = 100\left[10(A-a)+B-b\right]+10(T-t)+U-u = 100\cdot 31+31 = 3131 = n^2-m^2$$

Obviously, $3131 = 31\cdot 101$, and both 31 and 101 are prime numbers. So we have

$$(n-m)(n+m) = 31\cdot 101$$

This gives the pair of equations

$$\begin{aligned} n - m &= 31 \\ n + m &= 101 \end{aligned}$$

Adding the two equations gives $2n = 132 \Rightarrow n = 66$, and subtracting the top from the bottom gives $2m = 70 \Rightarrow m = 35$. Squaring m gives $m^2 = 1225$. Thus, Anne is currently 12 and Tom is currently 25.

Answer: Anne is 12 and Tom is 25.