BRITISH COLUMBIA SECONDARY SCHOOL MATHEMATICS CONTEST, 2013

Solutions

Junior Preliminary

1. Let b and g be the number of boys, including Danny, and girls in Danny's family. Then, $g = b - 1$ and $g = 2(g - 1)$. Hence, $g = 2$ and $b = 3$. So the total number of children, including Danny, in Danny's family is $b + g = 5$.

Answer is (C).

---

2. For each of the 24 ways there are to arrange the four students, there are five ways to place the fifth student among them. Hence, the total number of ways to arrange the five students is $5 \times 24 = 120$.

Answer is (E).

---

3. If there are 121 red lanterns then there are 180 green lanterns since for every two red lanterns there are three green lanterns. It follows that the 121^{st} red lantern must be in the $120 + 180 + 1 = 301^{st}$ position.

Answer is (D).

---

4. From the given information in one hour Pat can mow $\frac{1}{15}$ of one lawn, his brother can mow $\frac{1}{30}$ of one lawn, and his sister can mow $\frac{2}{15}$ of one lawn. It follows that, together, Pat's brother and sister can mow

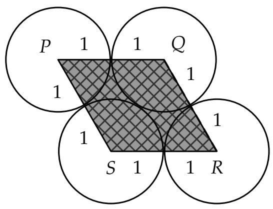
$$\frac{1}{30} + \frac{2}{15} = \frac{1}{6}$$

of a lawn in one hour. So it will take them 6 hours to mow one lawn when they work together.

Answer is (E).

---

5. The quadrilateral $PQRS$ is a rhombus with sides 2 that consists of two equilateral triangles, $\triangle PQS$ and $\triangle QRS$, each of side length 2. The area of each triangle is $\sqrt{3}$. Therefore, the rhombus has area $2\sqrt{3}$.



Answer is (A).

---

6. Since the largest of the given numbers, $\sqrt{n+3}$, must be the length of the hypotenuse of the triangle, Pythagoras' theorem implies $\left(\sqrt{n+3}\right)^2 = \left(\sqrt{n+1}\right)^2 + \left(\sqrt{n+2}\right)^2$, so $n = 0$, and then there is one such triangle.

Answer is (B).

7. Observe that 74° Rascal is

$$\frac{74-50}{110-50} = \frac{24}{60} = \frac{2}{5}$$

of the way from freezing to boiling. So the equivalent Melvin reading is

$$5 + \left(\frac{2}{5}\right)(45-5) = 5 + 16 = 21$$

Alternative solution:
Since both scales are linear, the data points $(5, 50)$ and $(45, 110)$ lead to the equation

$$R = \frac{3M}{2} + \frac{85}{2}$$

where R denotes Rascal degrees and M denotes Melvin degrees. Substituting the value $R = 74$ gives $M = 21$.

Answer is (C).

---

8. First observe that (c) requires that Alice and Emily are the same animal. If Emily is a frog, then she is lying and Alice is also a frog. If Emily is a moose, she is telling the truth and Alice is also a moose. Then, since Alice and Emily are the same animal, by (e) Dick is lying and so is a frog. Since Dick is a frog, by (b) Carla is telling the truth and so is a moose. Since Carla is a moose, by (d) Bob is lying and so is frog. Since Bob is a frog, by (a) Alice is lying and so Alice and Emily are frogs. Hence, there are four frogs. **Alternative solution:**
Alice is either a moose or a frog. Assume she is a moose. Then

- By (a) Bob is a moose.
- By (d) Carla is a frog.
- By (b) Dick is a moose.
- By (e) Emily is a frog.
- By (c) Alice is a frog. This is a contradiction, since the initial assumption was that Alice is a moose.

So Alice must be a frog. Then

- By (a) Bob is a frog.
- By (d) Carla is a moose.
- By (b) Dick is a frog.
- By (e) both Emily and Alice are frogs. Then (c) also shows that Alice is a frog. So there is not a contradiction.

Hence there are four frogs and one moose.

Answer is (D).

---

9. Note that for any real x for which none of the denominators in the expression is zero, it is true that

$$1 - \cfrac{1}{1 - \cfrac{1}{1 - \cfrac{1}{x}}} = 1 - \cfrac{1}{1 - \cfrac{x}{x-1}} = 1 - \cfrac{1}{\cfrac{x-1-x}{x-1}}$$

$$= 1 - \frac{x-1}{(-1)} = 1 + x - 1 = x$$

In the expression above $x = 2013$. Hence, its value is 2013.

Answer is (B).

10. Obviously neither a nor b can be large because the right-hand side is bounded above by 3000 and the left-hand side grows exponentially. Observe that the only powers of 9 to consider are $9^0 = 1$, $9^1 = 9$, $9^2 = 81$, and $9^3 = 729$. Since $9^4 = 6561$, the number $(2^a)\left(9^b\right) = 2a9b$ cannot be a power of 9, so a cannot be 0 and b must be even. Now consider cases:

- $b = 0$ gives a contradiction since $2^a = 2a90$ is impossible because no power of 2 ends in 0.
- $b = 2$ gives $(81)(2^a) = 2a92$, so that $2a92$ must be divisible by 9. This requires that $2 + a + 9 + 2 = a + 13$ be a multiple of 9. The only value of a in the required interval is $a = 5$. In this case, $2^5 \cdot 9^2 = 2592$. Thus $a = 5$, $b = 2$, and $a + b = 7$.

Answer is (C).

---

11. If a and b are single digit positive integers with $a > b$ that satisfy the required condition, their average is

$$\frac{a+b}{2} = b + \frac{a}{10} \Rightarrow 5a + 5b = 10b + a \Rightarrow 4a = 5b$$

The only single digit positive integers that satisfy this equation are $a = 5$ and $b = 4$, which satisfy (A), (B), and (D) above. If a and b are two digit positive integers with $a > b$ that satisfy the required condition, their average is

$$\frac{a+b}{2} = b + \frac{a}{100} \Rightarrow 49a = 50b$$

The only two digit positive integers that satisfy this equation are $a = 50$ and $b = 49$, which only satifies (A) above. In general, if a and b are two n digit positive integers with $a > b$ that satisfy the required condition, their average is

$$\frac{a+b}{2} = b + \frac{a}{10^n} \Rightarrow \left(5 \times 10^{n-1} - 1\right) a = \left(5 \times 10^{n-1}\right) b$$

The only n digit positive integers that satisfy this equation are $a = 5 \times 10^{n-1}$ and $b = 5 \times 10^{n-1} - 1$. In all of these case, $a - b = 1$.

Answer is (A).

---

12. Since $PQRS$ is a parallelogram, if you make an exact copy of the post and place it so that R is at P and Q is at S, the resulting object will be a right-rectangular box with edge lengths $33 + 47 = DS + 56$. Therefore, $DS = 24$.

Alternative solution:

Since the quadrilateral $PQRS$ is a parallelogram

$$\overline{PQ} = \overline{SR} \Rightarrow \overline{PQ}^2 = \overline{SR}^2$$

Therefore, Pythagoras' theorem gives

$$(56 - 33)^2 + 11^2 = \left(47 - \overline{DS}\right)^2 + 11^2$$

Solving for $\overline{DS}$ gives $\overline{DS} = 47 \pm 23 = 70$ or 24. Since $\overline{DS} < \overline{BQ} = 56$, the value is $\overline{DS} = 24$.

Answer is (A).

Senior Preliminary

1. Factoring 70 into prime factors gives

$$70 = 2 \times 5 \times 7$$

Therefore, the side lengths of the box are 3, 11, and 61. If the box had a top, its surface area would be

$$2\left[(2\,\text{cm}) \times (5\,\text{cm}) + (2\,\text{cm}) \times (7\,\text{cm}) + (5\,\text{cm}) \times (7\,\text{cm})\right] = 118\,\text{cm}^2$$

To obtain the box with no top with the largest possible surface area, remove one of the face with the smallest area, which is one of the 2×5 faces. This gives a surface area of

$$118\,\text{cm}^2 - 10\,\text{cm}^2 = 108\,\text{cm}^2$$

Answer is (B).

---

2. Given that $x^2 + x = 1$:

$$x^3 + 2x^2 + 2013 = x\left(x^2 + x\right) + x^2 + 2013 = x^2 + x + 2013 = 1 + 2013 = 2014$$

Answer is (E).

---

3. Solve the equation for x in terms of y go give

$$x = \frac{763 - 3y}{2}$$

In order for x to be a positive integer, $763 - 3y$ must be a positive multiple of 2. The smallest positive integer y for which this is the case is $y = 1$. In fact, $763 - 3y$ is a multiple of 2 if y is odd. Hence, let $y = 2k - 1$, where k is a positive integer. This gives

$$x = \frac{763 - 3y}{2} = 383 - 3k$$

This is positive for

$$k < \frac{383}{3} = 127\tfrac{2}{3}$$

So the largest possible value of k is 127. Therefore, the values of k that give positive integer values of both x and y are $1 \le k \le 127$. This gives a total of 127 ordered pairs (x, y).

Answer is (D).

---

4. Since the platform rotates at one revolution per hour, it takes 3600 seconds to make one complete revolution. Hence, the rotation rate is $\dfrac{360}{3600} = 0.1\,\text{deg/sec}$ and the number of degrees the platform rotates between firings is 27.5. If k is the number of firings until the ray is pointed in the same direction, then $27.5\,(k-1) = 360n$ for some integer n. Multiplying by 10 and cancelling common factors gives

$$275\,(k-1) = 3600n \Rightarrow 11\,(k-1) = 144n$$

Therefore, the smallest values of k and n that give integer solutions are $k = 145$ and $n = 11$. So, the ray must be fired 145 times before the same spot is hit for the second time. In this time, the platform has rotated 11 times.

Answer is (D).

5. Albert should divide the sixteen coins into two groups of six coins and one group of four coins. He then uses the following procedure:

   - Weigh the two groups of six coins. This is the first weighing.
   - If one of the groups of six is heavier than the other, then divide the heavier group into two groups of three coins and weigh the two groups of three. One of the two groups will be heavier than the other. This is the second weighing.
   - Weigh two of the coins in the heavier group. If one is heavier, the heavy coin has been found. If neither is heavier, then the third coin in the group is the heavy coin. This is the third weighing.
   - If neither of the groups of six coins is heavier than the other, the heavy coin is in the group of four, and there has only been one weighing so far.
   - Divide the group of four into two groups of two coins and weigh them. This is now the second weighing.
   - Weigh the two coins in the heavier group to identify the heavy coin. This is now the third weighing.

   In any of the cases above, at most three weighings were required.

   Answer is (C).

---

6. The amount the gambler has at each stage follows the sequence

$$k \to k-5 \to 2\,(k-5) \to 2\,(k-5)-5 \to 2\,(k-5)-5-5 \to 2\,[2\,(k-5)-5-5]-5 = 0$$

   solving for k gives $k = 11.25$.

   Alternative solution:
   Working backwards from the gambler's last expenditure, the amount he has just **before** his next expenditure or gambling session is

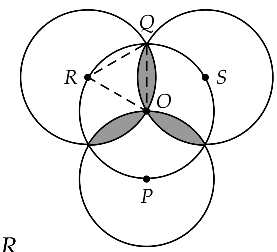
$$5 \to 2.50 \to 7.50 \to 12.50 \to 6.25 \to 11.25$$

   So the gambler had \$11.25 at the beginning of his day of gambling.

   Answer is (B).

---

7. Let Q be the intersection point of the upper three circles. Triangle ORQ is equilateral so that



$$\angle ORQ = \frac{\pi}{3} \text{ radians}$$

   The area of one half one of the three petals is

$$\begin{aligned} A_{\text{half petal}} &= \text{area of circular sector } OQR - \text{area of } \triangle OQR \\ &= \frac{1}{2}\left(\frac{\pi}{3}\right)(1)^2 - \frac{1}{2}(1)\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6} - \frac{\sqrt{3}}{4} \end{aligned}$$

   So the area of the shaded region is

$$6 \times A_{\text{half petal}} = 6\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right) = \pi - \frac{3\sqrt{3}}{2} = \frac{2\pi - 3\sqrt{3}}{2}$$

   Answer is (A).

8. From the definition of absolute value

$$||5x-4|-6| = 3 \Rightarrow |5x-4| - 6 = \pm 3 \Rightarrow |5x-4| = 9 \text{ or } 3$$

So that $5x - 4 = \pm 9$ or ± 3. Hence, there are four values of x that satisfy the equation.

Answer is (E).

---

9. Let x, y, and z be the price of the dictionary, pencil, and eraser, respectively, where $z < y < x$. Then the given information requires that

$$x + y + z = 70.35 \text{ and } xyz = 70.35$$

Factoring the integer 7035 into prime factors gives $7035 = 3 \times 5 \times 7$. Hence,

$$xyz = 3 \times 5 \times 7 \times 67 \times 10^{-2}$$

Since there are only three numbers, two of these prime factors must be multiplied to form one of x, y, or z. There are ${}_4C_2 = 6$ ways to choose the two numbers to multiply. This gives the following possibilities

$$\begin{aligned}
3 \times 5 \times (7 \times 67) &= 3 \times 5 \times 469 \\
3 \times 7 \times (5 \times 67) &= 3 \times 7 \times 335 \\
5 \times 7 \times (3 \times 67) &= 5 \times 7 \times 201 \\
3 \times (5 \times 7) \times 67 &= 3 \times 35 \times 67 \\
5 \times (3 \times 7) \times 67 &= 5 \times 21 \times 67 \\
7 \times (3 \times 5) \times 67 &= 7 \times 15 \times 67
\end{aligned}$$

Distributing the 10^{-2} among the factors gives possible values for the variables x, y, and z. The requirement that the sum is 70.35 gives a unique set of values, since

$$67 + 3 + 35 \times 10^{-2} = 70.35 \Rightarrow x = 67, \ y = 3, \text{ and } z = 0.35$$

Testing other possibilities shows that this is the only combination that gives the required sum. For example,

$$\begin{aligned}
201 \times 10^{-1} + 5 \times 10^{1} + 7 \times 10^{-2} &= 70.17 \\
67 + 21 \times 10^{-1} + 5 \times 10^{-1} &= 69.60 \\
469 \times 10^{-1} + 5 + 3 \times 10^{-1} &= 52.20 \\
469 \times 10^{-2} + 5 \times 10^{1} + 3 \times 10^{-1} &= 54.99
\end{aligned}$$

So the cost of the pencil, in dollars, is $y = 3.00$.

Answer is (D).

10. Let X denote the left-hand vertex of the triangle formed on the front face of the aquariums when it is tilted. Since the volume and width are fixed, and the tilted-base area is $\frac{4}{5}$ the area of the original base, the length AX must be $\frac{4}{5} \times 30 = 24$. Hence, the volume of the water in the aquarium is

$$20 \times \text{Area of triangle } ADX = 20 \times \frac{1}{2} \times 24 \times 15 = 3600$$

Therefore, the depth d of the water when the aquarium is level is given by

$$30 \times 20 \times d = 600d = 3600 \Rightarrow d = 6$$

Answer is (B).

---

11. Let p be a prime number and suppose that $2p+1$ is a perfect cube. Then there is an integer x such that

$$2p+1 = x^3 \Leftrightarrow x^3 - 1 = 2p \Leftrightarrow (x-1)\left(x^2+x+1\right) = 2p$$

Observe that

$$x^2 + 2 > 0 \Longrightarrow x^2 + x + 1 > x - 1$$

so that $x-1$ must be the smaller of the two factors of $2p$. Therefore, since any integer, in particular $2p$, can be factored into the product of primes in only one way, it must be true that

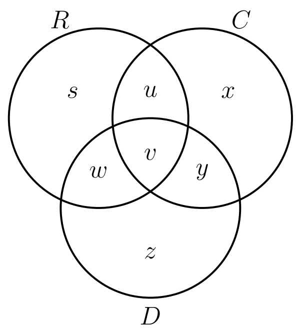
$$x - 1 = 2 \text{ and } x^2 + x + 1 = p \Longrightarrow x = 3 \text{ and } p = 13$$

Since $p = 13$ is the only solution, there is exactly one prime number p for which $2p+1$ is a perfect cube.

Answer is (B).

---

12. See the Venn diagram at the right, in which R is the set of days when it was rainy, C is the set of days it was cold, and D is the set of days when it was cloudy. From the given information it is clear that $z = 0$ and $s + u = 0$. Since all of the values must be nonnegative, this means that $z = s = u = 0$. Further, $x + u = 20$, so that $x = 20$. The remaining information gives



$$\begin{aligned} y + v &= 70 \\ y + v + w &= 80 \\ v + w &= 75 \end{aligned}$$

Subtracting the third equation from the second gives $y = 5$. Substituting into the first equation gives $v = 65$.

Answer is (D).

Junior Final, Part A

1. With two flags Amanda can make the following signals: RR, RW, and WR. With three flags she can make WRR, RWR, RRW. There are six possible signals.

Answer is (C).

2. Consider the squares $ABCD$ and $EFGH$ as described. The triangle AEH is a right-angle triangle with $\angle A = 90^\circ$ and $AE = 7AH$. Thus, the area of the triangle $AEH = \frac{7}{2}(AH)^2$. Since there are four identical triangles, the area of the square $EFGH$ is given by the area of $ABCD$ minus $14(AH)^2$. On the other hand, the area of the square $ABCD$ is given by $(AB)^2 = (8AH)^2 = 64(AH)^2$. It follows that the ratio of the area of $EFGH$ to the area of $ABCD$ is given by

$$\frac{64(AH)^2 - 14(AH)^2}{64(AH)^2} = \frac{50}{64} = \frac{25}{32}$$

Alternative solution:
Let $AB = 8$. Then $AE = 7$ and $AH = 1$. By Pythagoras' theorem

$$HE = \sqrt{AE^2 + AH^2} = \sqrt{49+1} = \sqrt{50}$$

Hence, the area of square $EFGH$ is $\left(\sqrt{50}\right)^2 = 50$ and the area of square $ABCD$ is $8^2 = 64$. Therefore, the fraction of the area of square $ABCD$ that is contained in the square $EFGH$ is

$$\frac{50}{64} = \frac{25}{32}$$

Answer is (C).

---

3. Let x be the amount you have to pay at the first door. At the second door you pay $2x+1$. At the third door you pay $2(2x+1)+1 = 4x+3$. At the fourth door you pay $2(4x+3)+1 = 8x+7$. In all you pay

$$x + 2x + 1 + 4x + 3 + 8x + 7 = 15x + 11 = 56 \Rightarrow x = 3$$

Answer is (E).

---

4. Let x, y, and z be the number of candies which Nora gave to Jack, Mona, and Ian respectively. Since there are 120 candies altogether, $x + y + z = 120$. Since a total of \$7.00 = 700¢ is spent on the candies $5x + 6y + 7z = 700$. Multiplying the first equation by 6 and subtracting the second equation gives $n = x - z = 20$.

Answer is (A).

---

5. Work out the sum after one move, two moves, and so on, always recording only the larger sum:

9	2	6	3
1	4	5	7
7	3	2	5
2	8	4	1

→

9	2	6	3
1	4	5	7
9	3	2	5
	10	4	1

→

9	2	6	3
10	4	5	7
	13	2	5
		14	1

→

19	2	6	3
	17	5	7
		16	5
			15

→

	21	6	3
		22	7
			21

→

		28	3
			29

→

			32

Answer is (C).

6. Make a table showing the truth values for each of the statements given that the money is behind each of the doors.

money	door 1	door 2	door 3	door 4	door 5
1	false	true	false	false	false
2	true	false	false	false	false
3	true	false	true	true	true
4	true	false	false	false	false
5	true	false	false	true	false

If exactly two of the signs are true, then the money must be behind door 5.

Answer is (E).

---

7. For $i = 1, 2, 3$, denote the time, distance and velocity in time period i by t_i, d_i, and v_i respectively. The definition speed $= \dfrac{\text{distance}}{\text{time}}$ then gives the total time taken as

$$t_1 + t_2 + t_3 = \frac{d_1}{v_1} + \frac{d_2}{v_2} + \frac{d_3}{v_3} = \frac{30}{v} + \frac{20}{2v} + \frac{15}{3v} = \frac{45}{v} \Rightarrow \overline{v} = \frac{30 + 20 + 15}{\left(\dfrac{45}{v}\right)} = \frac{13v}{9}$$

Answer is (B).

---

8. Form a table with the given information:

	large	small	total
red	22		
green			26
total		12	50

$\rightarrow$

	large	small	total
red	22		24
green			26
total	38	12	50

$\rightarrow$

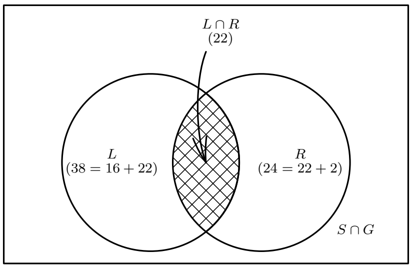
	large	small	total
red	22	2	24
green	16	10	26
total	38	12	50

There are 10 small green balls.

Alternative solution:

This problem can be solved using the inclusion-exclusion principle. Let L be the set of large balls, S the set of small balls, R the set of red balls, and G the set of green balls. From the given information

$$n(S) = 12,\ n(G) = 26, \text{ and } n(L \cap R) = 22$$



Here, $n(S)$ represents the number of elements (balls in this case) in the set S. Since the sets S and L make up all of the balls, as do R and G, the number of balls in L and R is given by

$$n(L) = 50 - n(S) = 38 \text{ and } n(R) = 50 - n(G) = 24$$

See the Venn diagram at the right. The number of balls that are either large or red or both is given by

$$n(L \cup R) = n(L) + n(R) - n(L \cap R) = 38 + 24 - 22 = 40$$

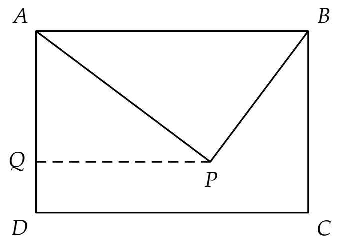
This value can also be found as $16 + 22 + 2 = 40$. This is the sum of the number of balls that are large and green, large and red, and small and red, which are represented by the three disjoint regions in the Venn diagram. In either case, the number of balls that are both small and green is

$$n(S \cap G) = 50 - n(L \cup R) = 10$$

Answer is (D).

9. There was 30 minutes between when the plane landed and when the mail truck met the motorcycle. If the truck had continued on its way, it would have traveled for 10 minutes, picked up the mail when the plane was supposed to land, and taken 10 minutes to return to where it met the motorcycle, accounting for the 20 minutes which the truck was early. But the motorcycle traveled for 30 minutes in addition to the 10 minutes which the truck would have traveled to meet the plane if it had landed on time, so the plane must have been 40 minutes early.

Answer is (B).



10. First note that $12^2 + 9^2 = 225 = 15^2$, so that

$$AB^2 = AP^2 + BP^2$$

which means that APB is a right triangle. Draw PQ perpendicular to AD on the diagram above. Then $\triangle AQP$ is similar to $\triangle BPA$ so that

$$\frac{AQ}{BP} = \frac{AP}{BA} \quad \Rightarrow \quad \frac{AQ}{9} = \frac{12}{15} \Rightarrow AQ = \frac{36}{5}$$
$$\frac{PQ}{AP} = \frac{AP}{BA} \quad \Rightarrow \quad \frac{PQ}{12} = \frac{12}{15} \Rightarrow PQ = \frac{48}{5}$$

Hence

$$DQ = 10 - AQ = \frac{14}{5} \Rightarrow DP = \sqrt{DQ^2 + PQ^2} = \sqrt{\frac{14^2 + 48^2}{25}} = 10$$

Answer is (D).

Junior Final, Part B

1. Using Pythagoras' theorem the altitudes of the two triangles are

$$\sqrt{25^2 - 20^2} = 15 \text{ and } \sqrt{25^2 - 15^2} = 20$$

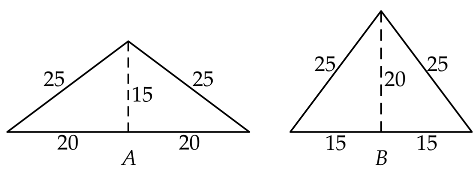
Hence, the areas of the two triangles are

$$A = 20 \times 15 = 300$$

and

$$B = 15 \times 20 = 300$$

so $A = B = 300$.



Answer: The relation is $A = B$.

2. Let x and y be positive integers satisfying the relation $x^2 - y^2 = 140$, then we have $(x+y)(x-y) = 140$. Let $a = x + y$ and $b = x - y$. Then

$$a + b = (x+y) + (x-y) = 2x \text{ and } a - b = (x+y) - (x-y) = 2y$$

and therefore, both $a + b$ and $a - b$ are even. It follows that a and b have the same parity; that is, both are even or both are odd. Note also that $x = \dfrac{a+b}{2}$ and $y = \dfrac{a-b}{2}$. Now

$$x^2 - y^2 = (x+y)(x-y) = ab = 140$$

so that both a and b are factors of 140. The possible factoring of 140 and corresponding parity are given in the table below:

a	b	Same Parity
140	1	No
70	2	Yes
35	4	No
28	5	No
20	7	No
14	10	Yes

This table shows that (a, b) is either $(70, 2)$ or $(14, 10)$. It follows that $(x, y) = \left(\dfrac{a+b}{2}, \dfrac{a-b}{2}\right)$ is either $(36, 34)$ or $(12, 2)$.

Answer: The only two points are $(36, 34)$ and $(12, 2)$.

---

3. (a) Just do the calculation:

$$3^3 + 7^3 + 1^3 = 27 + 343 + 1 = 371$$

Answer: See calculation above.

(b) From the calculation above it is clear that

$$3^3 + 7^3 + 0^3 = 27 + 343 + 0 = 370$$

is an Armstrong number. Further, since $7^3 = 343$ and $4^3 = 64$, another Armstrong number is

$$4^3 + 0^3 + 7^3 = 64 + 0 + 343 = 407$$

The last fourth three digit Armstrong number is obtained by noting that $5^3 = 125$, $3^3 = 27$, and $1^3 = 1$, so that

$$1^3 + 5^3 + 3^3 = 1 + 125 + 27 = 153$$

Answer: Two of the three numbers 370, 407, and 153.

Problem 3 continued ...

...Problem 3 continued

(c) Let a be a single digit from 1 to 9 and b a single digit from 0 to 9. It is required to find a and b such that $a^2 + b^2 = 10a + b$, or to find integer solutions for the quadratic equation $b^2 - b + a(a - 10) = 0$ where a is a positive single digit integer. The discriminant for this quadratic equation is given by

$$(-1)^2 - 4 \times 1 \times a\,(a - 10) = 1 + 4a\,(10 - a)$$

For the solution to be an integer, the discriminant must be a perfect square (though this is not enough to guarantee an integer solution). The table below shows the value of the discriminant for all possible values of a.

a	$D = 1 + 4a(10 - a)$	Perfect Square
1	$1 + 4 \times 1 \times (10 - 1) = 1 + 4 \times 1 \times 9 = 37$	No
2	$1 + 4 \times 2 \times (10 - 2) = 1 + 4 \times 2 \times 8 = 65$	No
3	$1 + 4 \times 3 \times (10 - 3) = 1 + 4 \times 3 \times 7 = 85$	No
4	$1 + 4 \times 4 \times (10 - 4) = 1 + 4 \times 4 \times 6 = 97$	No
5	$1 + 4 \times 5 \times (10 - 5) = 1 + 4 \times 5 \times 5 = 101$	No
6	$1 + 4 \times 6 \times (10 - 6) = 1 + 4 \times 6 \times 4 = 97$	No
7	$1 + 4 \times 7 \times (10 - 7) = 1 + 4 \times 7 \times 3 = 85$	No
8	$1 + 4 \times 8 \times (10 - 8) = 1 + 4 \times 8 \times 2 = 65$	No
9	$1 + 4 \times 9 \times (10 - 9) = 1 + 4 \times 9 \times 1 = 37$	No

Since none of the values above is a perfect square, there are no two digit Armstrong numbers.

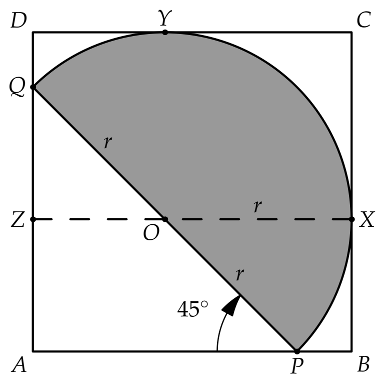
Answer: See proof above.

---

4. See Senior Final Part B, Problem 1(a).

---

5. Let r be the radius of the semicircle. Since PQ is a diameter of the semicircle, the centre O of the semicircle is at the midpoint of PQ and



$$OP = OQ = OX = r$$

Further, since the semicircle is tangent to the sides of the square, the line OX is perpendicular to side OC and so, is parallel to side AB. Extend the line segment OX to intersect side AD at point Z. Now OQZ is a 45°-45°-90° triangle with hypotenuse r. Hence,

$$OZ = \frac{r}{\sqrt{2}}$$

This gives

$$1 = XZ = OZ + OX = \frac{r}{\sqrt{2}} + r \Rightarrow r = \frac{1}{1 + \frac{1}{\sqrt{2}}} = 2 - \sqrt{2}$$

Therefore, the area of the semicircle is

$$\text{Area} = \frac{1}{2}\pi \left(2 - \sqrt{2}\right)^2 = \pi \left(3 - 2\sqrt{2}\right)$$

Answer: The area of the semicircle is $\pi \left(3 - 2\sqrt{2}\right)$

Senior Final, Part A

1. The maximum speed of the train is given by $f(x) = 120 - k\sqrt{x}$ where x is the number of cars and k is the constant of proportionality. With the given information the value of the k is obtained as

$$f(4) = 120 - 2k = 90 \Rightarrow k = 15$$

Hence, $f(x) = 120 - 15\sqrt{x}$. So the largest number of cars that the engine can move is given by the equation

$$f(x) = 120 - 15\sqrt{x} = 0 \Rightarrow \sqrt{x} = \frac{120}{15} = 8 \Rightarrow x = 64$$

But if there are 64 cars the train does not move, so the largest number of cars which the train can move is 63.

Answer is (B).

---

2. From the definition of absolute value

$$||5x - 4| - 6| = 6 \Rightarrow |5x - 4| - 6 = \pm 6 \Rightarrow |5x - 4| = 0 \text{ or } 12$$

So that $5x - 4 = 0$ or ± 12. Hence, there are three values of x that satisfy the equation.

Answer is (D).

---

3. See Junior Final Part A, Problem 5.

Answer is (C).

---

4. For each number

$$\Pr(\text{positive}) = \frac{9}{30} = \frac{3}{10} \quad \text{and} \quad \Pr(\text{negative}) = \frac{20}{30} = \frac{2}{3}$$

Hence,

$$\begin{aligned} \Pr(\text{positive product}) &= \Pr(\text{both positive}) + \Pr(\text{both negative}) \\ &= \Pr(\text{positive})^2 + \Pr(\text{negative})^2 = \frac{9}{100} + \frac{4}{9} = \frac{481}{900} \end{aligned}$$

Answer is (D).

5. Let v be the velocity of the mosquito, t_1 be the time it takes the mosquito to go from the slower train to the faster, and t_2 the total time it takes the mosquito to make the return trip. Then t_1 satisfies the equation

$$vt_1 = 3 - 5t_1 \Rightarrow t_1 = \frac{3}{v+5}$$

Measuring distances from the point where the mosquito initially leaves the slower train, the time t_2 satisfies the equation

$$vt_1 - v(t_2 - t_1) = 3t_2 \Rightarrow \frac{6v}{v+5} - vt_2 = 3t_2 \Rightarrow t_2 = \frac{6v}{(v+5)(v+3)}$$

During the time t_2 the two trains are approaching each other at a velocity of $3 + 5 = 8$ kilometres per hour. In this time, the distance between the trains has decreased from 3 to 1 kilometre, a distance of 2 kilometres. Hence, the velocity v satisfies the equation:

$$8t_2 = 2 \Rightarrow 8\left(\frac{6v}{(v+5)(v+3)}\right) = 2 \Rightarrow 24v = (v+5)(v+3)$$

Multiplying out in the equation above, and then rearranging and factoring gives:

$$24v = v^2 + 8v + 15 \Rightarrow v^2 - 16v + 15 = 0 \Rightarrow (v-15)(v-1) = 0$$

Two solutions to the quadratic equation above are $v = 15$ and $v = 1$. Obviously, the mosquito must go faster than both trains, so the only possible solution is $v = 15$.

Answer is (C).

---

6. First calculate the volume of the unit tetrahedron and then use the distance from a vertex to the centroid to calculate the volume of the tetrahedron inscribed in the unit sphere.
Place the vertices of the unit equilateral triangle at $(0,0)$, $(1,0)$, and $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. The area of this triangle is $\frac{\sqrt{3}}{4}$. The centroid of this triangle is at $\left(\frac{1}{2}, \frac{\sqrt{3}}{6}\right)$. The fourth vertex of the tetrahedron is above this centroid at $\left(\frac{1}{2}, \frac{\sqrt{3}}{6}, z\right)$ and must be a distance 1 from the origin:

$$\frac{1}{4} + \frac{3}{36} + z^2 = 1 \Rightarrow z^2 = 1 - \frac{1}{4} - \frac{1}{12} = \frac{2}{3} \Rightarrow z = \sqrt{\frac{2}{3}}$$

The volume of this tetrahedron is

$$\text{Volume} = \frac{1}{3}Bh = \frac{1}{3}\frac{\sqrt{3}}{4}\sqrt{\frac{2}{3}} = \frac{\sqrt{2}}{12}$$

The centroid of the tetrahedron is at $\left(\frac{1}{2}, \frac{\sqrt{3}}{6}, \frac{\sqrt{6}}{12}\right)$ so the distance from the centroid to the vertex at the origin is

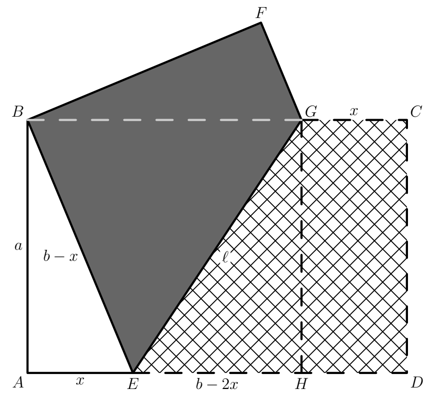
$$\sqrt{\frac{1}{4} + \frac{3}{36} + \frac{6}{144}} = \sqrt{\frac{1}{4} + \frac{1}{12} + \frac{1}{24}} = \sqrt{\frac{3}{8}}$$

The volume of the tetrahedron inscribed in the unit sphere is

$$\frac{\frac{\sqrt{2}}{12}}{\left(\sqrt{\frac{3}{8}}\right)^3} = \frac{\sqrt{2}}{12}\frac{16\sqrt{2}}{3\sqrt{3}} = \frac{8\sqrt{3}}{27}$$

Answer is (A).

7. Let $ABCD$ be the rectangle and EG the crease. Let a be the length of shorter side of the paper and b the length of the longer side, let x be the distance of the lower fold point from the vertex E, and let $\ell = |\overline{EG}|$ be the length of the crease. Since sides ED and EB are the same part of the lower part of the paper, $|\overline{EB}| = b - x$. Then applying Pythagoras' theorem to the triangle ABE gives



$$(b-x)^2 = a^2 + x^2 \Rightarrow b^2 - 2bx + x^2 = a^2 + x^2$$
$$\Rightarrow 2bx = b^2 - a^2 \Rightarrow x = \frac{b^2 - a^2}{2b}$$

Triangles ABE and BFG are congruent, so that $|\overline{GC}| = x$. Hence, $|\overline{E}| = b - 2x$. Using the expression above for x gives

$$b - 2x = b - \frac{b^2 - a^2}{b} = \frac{a^2}{b}$$

Applying Pythagoras' theorem to triangle EGH gives

$$\ell^2 = a^2 + \left(\frac{a^2}{b}\right)^2 = \frac{a^2b^2 + a^4}{b^2}$$

If $\ell = b$, then this equation gives

$$b^2 = \frac{a^2b^2 + a^4}{b^2} \Rightarrow b^4 - a^2b^2 - a^4 = 0$$

Treating this as a quadratic equation in b^2 gives

$$b^2 = \frac{a^2 \pm \sqrt{a^4 + 4a^4}}{2} = a^2\left(\frac{1 \pm \sqrt{5}}{2}\right)$$

Take the $+$ to give a positive value of b^2. Dividing by a^2 and taking the square root gives the ratio of the longer side of the paper to the shorter side as

$$\frac{b}{a} = \sqrt{\frac{1+\sqrt{5}}{2}}$$

Answer is (E).

8. Let a be the length of the side of the larger tile and b be the length of the side of the smaller tile. Then

$$na^2 = (n+76)\,b^2 \Rightarrow n\left(a^2 - b^2\right) = 76b^2 \Rightarrow n\,(a-b)\,(a+b) = 2^2 \times 19b^2$$

Since a and b have no common factors, then $a+b$ and $a-b$ also have no factors in common with b. Hence, the product $(a-b)\,(a+b)$ must be a factor of $2^2 \times 19$. The only two possibilities give integer solutions for a and b are

$$\left.\begin{array}{rcl} a+b & = & 19 \\ a-b & = & 1 \end{array}\right\} \Rightarrow a = 10,\ b = 9,\ \text{and } n = 324$$

or

$$\left.\begin{array}{rcl} a+b & = & 38 \\ a-b & = & 2 \end{array}\right\} \Rightarrow a = 20,\ b = 18,\ \text{and } n = 324$$

There must be 324 tiles.

Answer is (A).

---

9. Letting $f(z) = z^2 + 13z + 3 = n^2$ for some integer n and solving for z gives:

$$z = \frac{-13 \pm \sqrt{169 - 4\left(3 - n^2\right)}}{2} = \frac{-13 \pm \sqrt{157 + 4n^2}}{2}$$

In order for z to be an integer it is required that $157 + 4n^2$ is a perfect square. This gives

$$157 + 4n^2 = m^2 \Rightarrow m^2 - 4n^2 = (m - 2n)\,(m + 2n) = 157$$

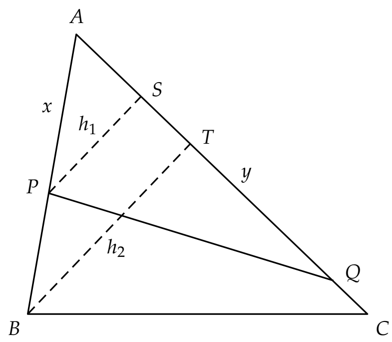
Since only n^2 and m^2 appear in the equations above, there is no loss of generality to assume that $n > 0$ and $m > 0$. In this case it must be true that $m > 2n$. Since 157 is prime, the only possible factoring is

$$m - 2n = 1 \text{ and } m + 2n = 157 \Rightarrow m = 79 \text{ and } n = 39$$

The only possible positive value is $n = 39$. Since there is only one possible value of n, there are only two values of z that make the value of the function a perfect integral square, which are the two solutions to the same quadratic. Hence, the required sum is -13.

Answer is (B).

10. Let $|\overline{AP}| = x$ and $|\overline{AQ}| = y$. Then



$$x + y = \frac{1}{2}(5+6+7) = 9 \Rightarrow y = 9 - x$$

Letting h_1 and h_2 be the heights of the triangles APQ and ABC, respectively, the areas of the two triangles are:

$$\text{Area } \triangle ABC = \frac{1}{2} \times 7 \times h_2$$
$$\text{Area } \triangle APQ = \frac{1}{2} y h_1 = \frac{1}{2}\text{Area } \triangle ABC = \frac{7}{4} \times h_2$$

Since triangles APS and ABT are similar

$$\frac{h_2}{5} = \frac{h_1}{x} \Rightarrow h_1 = \frac{x h_2}{5}$$

Substituting for h_1 in the equation for the area of triangle APQ and solving for xy gives:

$$\frac{1}{2} y \left(\frac{x h_2}{5}\right) = \frac{7 h_2}{4} \Rightarrow xy = \frac{35}{2}$$

Then using the fact that $y = 9 - x$ gives

$$x\,(9-x) = \frac{35}{2} \Rightarrow 2x^2 - 18x + 35 = 0$$

Solving for x gives

$$x = \frac{18 \pm \sqrt{18^2 - 8\,(35)}}{2} = \frac{9 \pm \sqrt{11}}{2}$$

Then

$$|\overline{PB}| = 5 - x = \frac{1 \pm \sqrt{11}}{2}$$

Since $|\overline{PB}|$ must be positive, the final result is $|\overline{PB}| = \dfrac{1+\sqrt{11}}{2}$.

Answer is (B).

Senior Final, Part B

1. (a) Suppose that a is a single digit from 1 to 9 and b is a single digit from 0 to 9. It is required to show that $1000a + 100b + 10b + a$ is divisible by 11. Then

$$1000a + 100b + 10b + a = 1001a + 110b = 11(91a + 10b),$$

Therefore any number of the form "$abba$" is divisible by 11.

Answer: See proof above.

(b) See Junior Final Part B, Problem 3 (c).

Answer: See proof above.

2. First note that for $f(x) = x^5 - 2x^4 + 3x^3 - 3x^2 + 2x - 1$, the value of the function at $x = 1$ is $f(1) = 0$. Hence, by the Factor Theorem $x - 1$ is a factor. Dividing by $x - 1$ gives

$$x^5 - 2x^4 + 3x^3 - 3x^2 + 2x - 1 = (x-1)\left(x^4 - x^3 + 2x^2 - x + 1\right)$$

Now grouping the expression the second set of parentheses gives

$$\begin{aligned} x^5 - 2x^4 + 3x^3 - 3x^2 + 2x - 1 &= (x-1)\left(x^4 + 2x^2 + 1 - x^3 - x\right) \\ &= (x-1)\left[\left(x^2+1\right)^2 - x\left(x^2+1\right)\right] \\ &= (x-1)\left(x^2+1\right)\left(x^2+1-x\right) \\ &= (x-1)\left(x^2+1\right)\left(x^2-x+1\right) \end{aligned}$$

The quadratic expressions $x^2 + 1$ and $x^2 - x + 1$ are irreducible since neither of the quadratic equations $x^2 + 1 = 0$ or $x^2 - x + 1 = 0$ have real solutions. So the complete factoring is

$$x^5 - 2x^4 + 3x^3 - 3x^2 + 2x - 1 = (x-1)\left(x^2+1\right)\left(x^2-x+1\right)$$

Answer: The factoring is $x^5 - 2x^4 + 3x^3 - 3x^2 + 2x - 1 = \left(x^2+1\right)(x-1)\left(x^2-x+1\right)$**.**

---

3. Consider the circle $(x-5)^2 + (y-3)^2 = 25$ and the parabola with equation $y = k + a(x-h)^2$.

(a) The points where the circle intersects the x-axis are determined by setting $y = 0$ in the equation to give

$$\begin{aligned} (x-5)^2 + 9 = 25 &\Rightarrow (x-5)^2 = 16 \\ \Rightarrow x - 5 = \pm 4 &\Rightarrow x = 1 \text{ or } x = 9 \end{aligned}$$

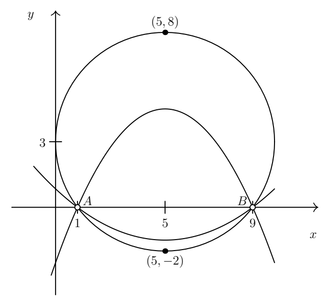
Thus, the two points are $A = (1, 0)$ and $B = (9, 0)$.

Answer: The points are $A = (1,0)$ **and** $B = (9,0)$**.**

(b) The line $x = h$ is the axis of symmetry of the parabola. the points A and B are symmetric about this line, so it goes through the midpoint of the line joining A and B. This point has x-coordinate $x = 5$. Therefore, $h = 5$.

Answer: The value of h **is** $h = 5$**.**

(c) The constant k is the y-coordinate of the vertex of the parabola. Hence, considering the diagram, which shows the circle and two possible parabolas, it appears that the parabola has only the points A and B if k satisfies $-2 < k < 8$.



Answer: The range of values of k **is** $-2 < k < 8$**.**

Problem 3 continued ...

... Problem 3 continued

(d) The equation of a parabola is of the form that has x-intercepts at $x = 1$ and $x = 9$ can be written as

$$y = a(x-1)(x-9)$$

where a is the same constant as in the given equation. Expanding and completing the square gives

$$y = a(x-1)(x-9) = a\left(x^2 - 10x + 9\right) = a\left[(x-5)^2 - 16\right] = a(x-5)^2 - 16a$$

Comparing this with the given equation shows that

$$k = -16a \Rightarrow a = -\frac{k}{16}$$

and the equation of the parabola can be written in terms of k alone as

$$y = -\frac{k}{16}(x-5)^2 + k = k\left[1 - \frac{1}{16}(x-5)^2\right]$$

Verification of solution:

Here is a verification of the range of possible k values. The points of intersection of the circle and the parabola $y = a\left(x^2 - 10x + 9\right)$ are given the equation

$$(x-5)^2 + \left(ax^2 - 10ax + 9a - 3\right)^2 = 25$$

Expanding gives the equation

$$a^2x^4 - 20a^2x^3 + \left[100a^2 + 2a(9-3)\right]x^2 - [10 + 20a(9a-3)]x + (9a-3)^2 = 0$$
$$a^2x^4 - 20a^2x^3 + \left[118a^2 - 6a + 1\right]x^2 - 10\left[18a^2 - 6a + 1\right]x + 81a^2 - 54a + 9 = 0$$

Since the circle and the parabola intersect the x-axis at $x = 1$ and $x = 9$ both $(x-1)$ and $(x-9)$ are factors of the expression above. Doing the appropriate polynomial divisions gives

$$a^2x^4 - 20a^2x^3 + \left(118a^2 - 6a + 1\right)x^2 - 10\left(18a^2 - 6a + 1\right)x + 81a^2 - 54a + 9 =$$
$$(x-1)(x-9)\left[a^2x^2 - 10a^2x + (3a-1)^2\right] = 0$$

If the parabola has points other than A and B in common with the circle, the quadratic equation $a^2x^2 - 10a^2x + (3a-1)^2 = 0$ has real roots. The solutions to this equation are given by

$$x = \frac{10a^2 \pm \sqrt{100a^4 - 4a^2(3a-1)^2}}{2a^2} = 5 \pm \frac{\sqrt{16a^2 + 6a - 1}}{a}$$

The solutions are real unless

$$16a^2 + 6a - 1 = (2a+1)(8a-1) < 0$$

This is the case if $-\frac{1}{2} < a < \frac{1}{8}$. Note that if $a = -\frac{1}{2}$ or $a = \frac{1}{8}$, then $x = 5$ is a solution, giving another point in common between the circle and parabola, so the strict inequalities are necessary.

Answer: The expression is $a = -\dfrac{k}{16}$ and the equation of the parabola is $k\left[1 - \frac{1}{16}(x-5)^2\right]$.

4. Let M be the number of men voters and W the number of women voters. Then,

$$\frac{M}{W} = \frac{a}{b}$$

so that $M = ka$ and $W = kb$ for some positive integer k. Using the fact that reducing the number of men by c and women by d gives the ratio $e : f$ leads to

$$\frac{M-c}{W-d} = \frac{e}{f} \Rightarrow \frac{ka-c}{kb-d} = \frac{e}{f} \Rightarrow (ka-c)\, f = (kb-d)\, e$$

Solving for k gives

$$k\,(af - be) = cf - de \Rightarrow k = \frac{cf-de}{af-be}$$

It follows that the total number of voters is

$$M + W = ka + kb = \left(\frac{cf-de}{af-be}\right)(a+b) = (a+b)\left(\frac{cf-de}{af-be}\right)$$

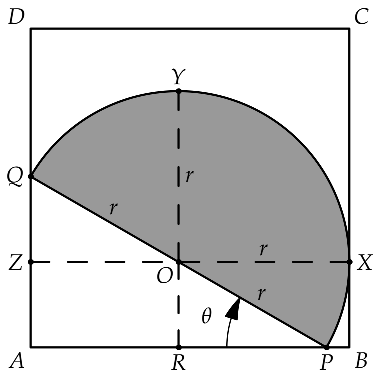
Answer: The total number of people who voted is $(a+b)\left(\dfrac{cf-de}{af-be}\right)$

---

5. Let r be the radius of the semicircle. Since PQ is a diameter of the semicircle, the centre O of the semicircle is at the midpoint of PQ and

$$\left|\overline{OP}\right| = \left|\overline{OQ}\right| = \left|\overline{OX}\right| = r$$

Further, since the semicircle is tangent to the sides of the square, the line OX is perpendicular to side BC and so, is parallel to side AB. Extend the line segment OX to intersect side AD at point Z. Drop a perpendicular from O to side AB, intersecting in point R. Extend this line to intersect the semicircle at Y. Now triangles OQZ and POR are congruent right triangle with hypotenuse r, and



$$\left|\overline{QZ}\right| = \left|\overline{OR}\right| = r\sin\theta \text{ and } \left|\overline{OZ}\right| = \left|\overline{PR}\right| = r\cos\theta$$

Hence,

$$1 = \left|\overline{XZ}\right| = \left|\overline{OZ}\right| + \left|\overline{OX}\right| = r\cos\theta + r \Rightarrow r = \frac{1}{1+\cos\theta}$$

Therefore, the area of the semicircle is

$$\text{Area} = \frac{1}{2}\pi\left(\frac{1}{1+\cos\theta}\right)^2$$

Since $\cos\theta$ decreases as θ increases from $\theta = 0$ to $\theta = \frac{\pi}{2}$, the area increases as θ increases. So the maximum area occurs when θ is as large as possible. The maximum possible value of θ that still guarantees that the semicircle still lies entirely inside the square is the θ for which the semicircle is tangent to side CD at point Y. This is the case when

$$r\sin\theta + r = 1 \Rightarrow \sin\theta = \frac{1-r}{r} = \frac{1-\left(\dfrac{1}{1+\cos\theta}\right)}{\left(\dfrac{1}{1+\cos\theta}\right)} = \cos\theta$$

Thus, the area is maximum when $\sin\theta = \cos\theta \Rightarrow \theta = 45^\circ$.

Answer: The area of the semicircle is $\dfrac{1}{2}\pi\left(\dfrac{1}{1+\cos\theta}\right)^2$ **and is maximum when** $\theta = 45^\circ$**.**