BRITISH COLUMBIA SECONDARY SCHOOL MATHEMATICS CONTEST, 2012

Solutions

Junior Preliminary

1. Dividing gives

$$\frac{2012}{3} = 670 + \frac{2}{3}$$

Hence, there are 670 positive integers less than 2012 that are divisible by 3. Half of them are even and half are odd. Therefore, there are 335 odd positive integers less than 2012 which are divisible by 3.

Alternative solution:

Note that the largest odd multiple of 3 less than 2012 is 2007 for which $2007 = 3 \times 669$. Further, since the k^{th} odd integer is $2k - 1$ and $669 = 2\,(335) - 1$, there are 335 odd positive integers less than 2012 and divisible by 3.

Answer is (B).

---

2. There are three possible positions for the B among the three letters, and once the B is positioned the A's fill the other two positions. So there are three ways to place the letters. The same argument shows that there are three ways to place the numbers. Hence there is a total of $3 \times 3 = 9$ possible license numbers.

Answer is (E).

---

3. If James selects six cookies, there could be two of each colour. If this is not the case, then he already has at least three cookies of the same colour. If not, when he selects one more cookie, it will make the third cookie of one of the three colours. Hence, the maximum number of cookies that James must select is seven.

Answer is (B).

---

4. A total of 15 cookies were purchased. When they are divided evenly among the three each gets 5 cookies. Hence, Mary gives up 1 cookie, and so gets \$1, and Anna gives up 4 cookies, and so gets \$4. Hence, the ratio of the money Abby gives to Anna to the money she gives to Mary is 4 : 1.

Answer is (A).

---

5. If x is the grade Eva got in each of the other two courses, then Eva's overall average for the full year is

$$\frac{5 \times 76 + 4 \times 80 + 90 + 2x}{12} = 80$$

Solving for x gives

$$2x = 960 - (5 \times 76 + 4 \times 80 + 90) = 170 \Longrightarrow x = 85$$

Note that since the overall average is 80% the four courses that averaged to 80% could be ignored and the twelve courses replaced by the remaining eight courses.

Answer is (A).

6. The following representations are unique: $4 = 2 + 2$, $6 = 3 + 3$, and $8 = 3 + 5$; while $10 = 3 + 7 = 5 + 5$, so 10 is the smallest even integer which can be represented as the sum of two primes in more than one way.

Answer is (D).

---

7. The area of a quarter disc of radius 2 is π, and the area of the triangle with the diagonal of the square as its hypotenuse is 2, so the area of the shaded region is $2 \times (\pi - 2) = 2\pi - 4$.

Answer is (D).

---

8. Any three consecutive integers have the form $a - 1$, a, and $a + 1$, for some integer a. Multiplying them and adding the middle number a gives

$$(a-1)(a)(a+1) + a = a(a-1)(a+1) + a = a\left(a^2 - 1\right) + a = a^3 - a + a = a^3$$

which is always a perfect cube.

Answer is (E).

---

9. Since ACD and BCE are congruent isosceles right triangles with right angles at C, triangle ABC is isosceles with $\left|\overline{AC}\right| = \left|\overline{BC}\right|$. So that, $\angle CBA = \angle CAB = 35°$. Hence, $\angle ACB = 180 - 70 = 110°$. Adding all of the angles around point C gives

$$\angle ACB + \angle ACD + \angle DCE + \angle ECB = 110° + 90° + \angle DCE + 90° = 360°$$

Solving gives $\angle DCE = 70°$.

Answer is (C).

---

10. Let x be the time from the end of one strike to the beginning of the following strike. When Big Ben strikes six o'clock there are six strikes and five intervals of length x between them. Since each strikes takes one quarter of a second, this gives the equation

$$5x + 6\left(\tfrac{1}{4}\right) = 5 \Longrightarrow 5x = 5 - \tfrac{3}{2} = \tfrac{7}{2} \Longrightarrow x = \tfrac{7}{10}$$

When Big Ben strikes twelve o'clock, there are twelve strikes and eleven intervals of length x between them. Hence, the total time is

$$11\left(\tfrac{7}{10}\right) + 12\left(\tfrac{1}{4}\right) = \tfrac{77}{10} + 3 = \tfrac{107}{10} = 10.7$$

Answer is (C).

11. First divide the 25 horses into 5 groups of 5 horses each and have each group run a race. Select the fastest horse of each of the 5 groups and have these horses run a race. The fastest horse in this race must be the fastest horse of the 25 horses. The next 2 horses in the group containing the fastest horse could be the second and third fastest horses, so select them. The 2 fastest horses in the group containing the second fastest horse could also be the second and third fastest horses, so select them. The fastest horse in the group containing the third fastest horse could be the third fastest horse, so select this horse. Now exactly 5 horses have been selected, and no other horse could be among the three fastest horses. So finally, run a race with the 5 selected horses and pick the fastest 2 horses as the second and third fastest horses. This gives a total of 7 races that must be run.

Answer is (C).

---

12. There are three cases to consider:

    i) Only one colour is used.
    The triangle can be painted in 4 different ways by choosing one of the 4 colours.

    ii) Exactly two colours are used.
    There are 4 ways to choose the colour that is used for the two edges and then 3 ways to choose the colour for the other edge. Then there are two possible colouring schemes: two long edges of the same colour (and the short edge being different), or one long edge and one short edge being the same colour (and the other long edge being different). This gives

    $$4 \times 3 \times 2 = 24$$

    ways to colour the triangle.

    iii) Exactly three colours are used.
    There are 4 ways of selecting the one colour that does not appear. Then with three colours, there are 3 choices with which to colour the short edge. (The long edges are interchangeable.) This gives

    $$4 \times 3 = 12$$

    ways to colour the triangle.

    In total, there are $4 + 24 + 12 = 40$ ways of colouring the triangle.

Answer is (B).

Senior Preliminary

1. If x and y are the grades she obtained in the other two courses, then Eva's overall average for the full year is

$$\frac{5 \times 75 + 4 \times 80 + 90 + x + y}{12} = 80$$

Solving for $x + y$ gives

$$x + y = 960 - (5 \times 75 + 4 \times 80 + 90) = 175$$

The highest percentage grade she could obtain in either course is 100%. Suppose that $x = 100$. Then $y = 175 - 100 = 75$. This is the minimum possible mark.

Answer is (C).

2. Lockers whose numbers are divisible by $8 = 2^3$ have a blue decal. Lockers whose numbers are divisible by

$$\text{lcm}\left(2^3, 2^2 \times 3\right) = 2^3 \times 3 = 24$$

have a blue and a yellow decal, where $\text{lcm}\,(8, 24)$ is the *lowest common multiple* of the two numbers 8 and 24, that is, it is the smallest number that is divisible by both numbers. Lockers whose numbers are a multiple of

$$\text{lcm}\left(2^3 \times 3, 3 \times 5\right) = 2^3 \times 3 \times 5 = 120$$

have blue, yellow, and green decals. So only lockers whose numbers are divisible by 120 have all three decal, and

$$\frac{750}{120} = \frac{25}{4} = 6\frac{1}{4}$$

So there are 6 lockers with all three decals.

Answer is (A).

---

3. Any four consecutive integers have the form a, $a+1$, $a+2$, and $a+3$ for some integer a. Multiplying them and adding 1 gives

$$a\,(a+1)\,(a+2)\,(a+3) + 1 = a^4 + 6a^3 + 11a^2 + 6a + 1 = \left(a^2 + 3a + 1\right)^2$$

which is always a perfect square.

Answer is (E).

---

4. Suppose there is a hand with less than 4 cards in each suit. Since there are 4 suits, this hand can have no more than 12 cards; a contradiction to the fact that each hand has 13 cards.

Answer is (D).

---

5. Represent the possible outcomes of rolling the two dice by an ordered pair (f, t), where f is the number on the fair die and t is the number on the trick die. The outcomes that give a sum of 7 when pair of dice is rolled are: $(1, 6)$, $(2, 5)$, $(3, 4)$, $(4, 3)$, $(5, 1)$, and $(6, 1)$. The probability of either $(1, 6)$ or $(6, 1)$ is $\frac{1}{6} \times \frac{1}{4}$. The probability of each of the other four possible outcomes is $\frac{1}{6} \times \frac{1}{8}$. So, the required probability is,

$$2 \times \left(\tfrac{1}{6} \times \tfrac{1}{4}\right) + 4 \times \left(\tfrac{1}{6} \times \tfrac{1}{8}\right) = \tfrac{1}{6}\left(\tfrac{1}{2} + \tfrac{1}{2}\right) = \tfrac{1}{6}$$

Observe that the trick die is irrelevant here, as it always lands on a positive integer less than 7. This means that it will always have a one in six chance of having the necessary *complement* of seven land on the standard die. For example, if the trick die lands on 4, there is a $\frac{1}{6}$ probability that the standard die lands on $7 - 4 = 3$.

Answer is (C).

6. Let $(0, -b)$ be the coordinates of the point where the line BC intersects the y-axis. By similar triangles,

$$\frac{a}{1+a} = \frac{b}{1} \Rightarrow b = \frac{a}{1+a}$$

Equating the areas of the regions on opposite sides of the y-axis gives:

$$\frac{1}{2}\left(1 + \frac{a}{1+a}\right)(a) = 1 + \frac{1}{2}\left(\frac{a}{1+a}\right)(1)$$

Simplifying gives

$$2a^2 - 2a - 2 = 0 \Rightarrow a^2 - a - 1 = 0$$

Using the quadratic formula gives

$$a = \frac{1 \pm \sqrt{1 - 4(1)(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

Since B lies in the third quadrant, $a > 0$. So take the plus in the solution above. Hence, $a = \dfrac{1+\sqrt{5}}{2}$.

Answer is (B).

---

7. Let x be the time from the end of one strike to the beginning of the following strike. When Big Ben strikes six o'clock there are six strikes and five intervals of length x between them. Since each strikes takes one quarter of a second, this gives the equation

$$5x + 6\left(\tfrac{1}{4}\right) = 5 \Longrightarrow 5x = 5 - \tfrac{3}{2} = \tfrac{7}{2} \Longrightarrow x = \tfrac{7}{10}$$

When Big Ben strikes twelve o'clock, there are twelve strikes and eleven intervals of length x between them. Hence, the total time is

$$11\left(\tfrac{7}{10}\right) + 12\left(\tfrac{1}{4}\right) = \tfrac{77}{10} + 3 = \tfrac{107}{10} = 10.7$$

Answer is (C).

---

8. Since the outside wheels are traveling twice as fast as the inside wheels, they travel twice as far in equal time. Since circumference is directly proportional to radius, it follows that the outside radius is twice the inside radius. But it is also 2 units more than the inside radius, and hence must be 4. It follows that the outside circumference is 8π.

Answer is (B).

---

9. Multiplying the second equation by 2 and subtracting the two equations gives $10xy = 10 \Rightarrow xy = 1$. Substituting $y = \dfrac{1}{x}$ in the second equation, multiplying by x, and re-arranging, gives the quadratic equation

$$2x^2 - 5x + 2 = 0 \Rightarrow (2x-1)(x-2) = 0$$

Solving gives two solutions $x_1 = \dfrac{1}{2}$ and $x_2 = 2$. Adding the solutions gives

$$x_1 + x_2 = \frac{1}{2} + 2 = \frac{5}{2}$$

Problem 9 continued ...

. . . Problem 9 continued

Alternative solution:

Writing the equation as

$$x^2 - \frac{5}{2}x + 1 = 0$$

and observing that if the quadratic equation $x^2 + bx + c$ has solutions x_1 and x_2, then the equation is

$$(x - x_1)(x - x_2) = x^2 - (x_1 + x_2)x + x_1x_2 = 0$$

So that the sum of the solutions is the negative of the coefficient of x in the quadratic equation, giving the same answer as above.

Answer is (D).

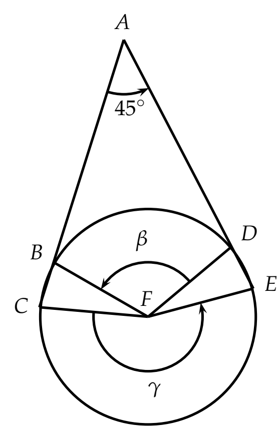
---

10. According to the predictions above, in the 100 years of the twenty-first century there will be 20 El Niño years. In 60% of these 20 years, that is 12 years, there will be a spring drought. In the remaining 80 years, there will be a spring drought in 10% of the years, that is, in 8 years. Hence, the total number of years in the twenty-first century having spring drought is predicted to be 20.

Answer is (C).

---

11. Let $\beta = \angle DFB$ and $\gamma = \angle CFE$. Since $|\overline{BC}| = |\overline{DE}|$ triangles BFC and EFD are congruent, so that



$$\angle BFC = \angle EFD = 25^\circ$$

Since triangle BFC is isosceles

$$\angle CBF = \tfrac{1}{2}(180^\circ - \angle BFC) = \tfrac{1}{2}(180^\circ - 25^\circ) = 77.5^\circ$$

Therefore, $\angle ABF = 180^\circ - 77.5^\circ = 102.5^\circ$. Similarly, $\angle ADF = 102.5^\circ$. Since $ABFD$ is a quadrilateral and $\angle BAD = \angle CAE = 45^\circ$, its interior angles add to 360°, so that

$$\angle BAD + \angle ABF + \angle ADF + \beta = 360^\circ$$
$$\Leftrightarrow \beta = 360^\circ - (45^\circ + 102.5^\circ + 102.5^\circ) = 110^\circ$$

Finally, adding the central angles of the circle gives

$$\angle BFC + \angle EFD + \beta + \gamma = 360^\circ$$
$$\Longrightarrow \gamma = 360^\circ - (25^\circ + 25^\circ + 110^\circ) = 200^\circ$$

which is the required angle.

Answer is (D).

12. The amount of yeast that Jake uses on the first day is $1000\,(1.3\times 0.25)$ and the amount that remains is $1000\,(1.3\times 0.75)$. The amount he uses on the second day is

$$1000\,(1.3\times 0.75)\,(1.3\times 0.25) = 1000\left(1.3^2\times 0.75\times 0.25\right)$$

and the amount remaining is $1000\left(1.3^2\times 0.75^2\right)$. Continuing is this ways gives the total amount of yeast used as

$$\begin{aligned}&1000\,(1.3\times 0.25)+1000\left(1.3^2\times 0.75.25\right)+1000\left(1.3^3\times 0.75^2\times 0.25\right)+\cdots\\&\qquad = 1000\,(1.3\times 0.25)\left[1+(1.3\times 0.75)+(1.3\times 0.75)^2+(1.3\times 0.75)^3+\cdots\right]\end{aligned}$$

Note that

$$1.3\times 0.75 = \left(\frac{13}{10}\right)\left(\frac{3}{4}\right) = \frac{39}{40} < 1$$

Hence, the geometric series above sums to a finite value given by

$$1000\,(1.3\times 0.25)\left(\frac{1}{1-\dfrac{39}{40}}\right) = 1300\,(0.25)\,(40) = 13000$$

Answer is (A).

Junior Final, Part A

1. Since the remainder when n is divided by 7 is 6, there is an integer k for which $n = 7k+6$. Therefore,

$$6n = 7\,(6k)+36 = 7\,(6k)+5\,(7)+1 = 7\,(6k+5)+1$$

Hence, when $6n$ is divided by 7, the remainder is 1.

Alternative solution:
Since the remainder when n is divided by 7 is 6,

$$n \equiv 6 \pmod 7$$

As a result

$$6n \equiv 36 \pmod 7 \equiv 1 \pmod 7$$

Hence, when $6n$ is divided by 7, the remainder is 1.

Answer is (E).

2. From the diagram it is clear that

$$2\triangle = 3\square \quad\text{and}\quad 3\bigcirc = 5\triangle$$

Therefore,

$$10\triangle = 15\square \quad\text{and}\quad 6\bigcirc = 10\triangle \Longrightarrow 6\bigcirc = 15\square \Longrightarrow 2\bigcirc = 5\square$$

So it takes 5 $\square$'s to balance $\bigcirc\ \bigcirc$.

Answer is (B).

3. Let n be the number of new members and o be the number of old members. Then $n + o = 500$. The ticket revenue is

$$n \times 14 + 0.7o \times 20 = 14n + 14o = 14(n + o) = 14 \times 500 = 7000$$

Answer is (A).

---

4. Let b be the length of the base of the trapezoid so the opposite edge has length $\frac{2}{3}b$, and let h be its height. If height of the lower triangle is h_1, then its area is $\frac{1}{2}bh_1 = 12$, and if the height of the upper triangle is h_2, then its area is $\left(\frac{1}{2}\right)\left(\frac{2}{3}\right)bh_2 = 10 \Rightarrow \frac{1}{2}bh_2 = 15$. Thus, the area of the trapezoid is

$$\frac{1}{2}h\left(b + \frac{2}{3}b\right) = \left(\frac{5}{3}\right)\left(\frac{1}{2}h_1 b + \frac{1}{2}h_2 b\right) = \frac{5}{3}(12 + 15) = 45$$

Therefore, the total area of the shaded region is $45 - 10 - 12 = 23$.

Answer is (B).

---

5. Using standard sum formulas gives

$$N = \frac{n(n+1)}{2} \text{ and } M = \frac{m(m+1)}{2}$$
$$\Rightarrow N - M = \frac{n^2 - m^2 + n - m}{2} = \frac{(n-m)(n+m+1)}{2} = 2012$$
$$\Rightarrow (n - m)(n + m + 1) = 4024 = 2^3 \times 503$$

Note that $n - m < n + m + 1$. If $n - m = a$ and $n + m + 1 = b$, then $n = \dfrac{a + b - 1}{2}$. Hence, $a + b$ must be odd, so that one must even and the other odd. The only two possibilities are $a = n - m = 1$ and $b = n + m + 1 = 4024$ or $a = n - m = 8$ and $bn + m + 1 = 503$. But, the requirement that $n > m + 1$ excludes the first possibility. Therefore,

$$n + m + 1 = 503 \Rightarrow n + m = 502$$

In fact $n = 255$, $m = 247$.

Answer is (D).

---

6. Listing the four digit numbers: 0122, 0212, 0221, 1022, 1202, 1220, 2012, 2021, 2102, 2120, 2201, and 2210. The seventh number is 2012.

Alternative solution:

Note that 2012 is the smallest number with 2 as the leading digit and that there are six numbers that do not have 2 as the leading digit. Therefore, 2012 is the seventh number.

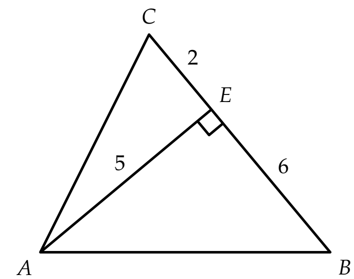
Answer is (B).

7. Taking side BC as base of triangle ABC, the altitude is AE. Since $\left|\overline{BC}\right| = 6 + 2 = 8$ and the area is 20, we have

$$\frac{1}{2}(8)\left|\overline{AE}\right| = 20 \Rightarrow \left|\overline{AE}\right| = 5$$

Applying Pythagoras' Theorem to the right triangle AEB gives

$$\left|\overline{AB}\right|^2 = \left|\overline{AE}\right|^2 + \left|\overline{EB}\right|^2 = 5^2 + 6^2 = 61 \Rightarrow \left|\overline{AB}\right| = \sqrt{61}$$



Answer is (C).

---

8. The minute hand travels 360° in 60 minutes, so it travels 6° in one minute. The hour hand travels 1/12 of a circle, or 30°, in 60 minutes, so it travels $\frac{1}{2}^{\circ}$ in one minute. Thus, the total number of degrees traveled by the minute hand in t minutes, after noon, is $m = 6t$, and the total number of degrees traveled by the hour hand is $h = t/2$. The minute and hour hand are perpendicular when the difference in the number of degrees traveled by the two hands is $180n - 90$, where n is a positive integer. This happens when

$$6t - \frac{t}{2} = \frac{11t}{2} = 90(2n - 1)$$

At midnight, after 12 hours, $t = 12 \times 60 = 720$, so the largest n that gives a time before midnight is given by

$$\frac{11(720)}{2} = 90(2n - 1) \Rightarrow 2n - 1 = 44 \Rightarrow 2n = 45$$

Thus, the last n before midnight is $n = 22$. Therefore, there are 22 times between noon and midnight when the minute and hour hands are perpendicular.

Answer is (D).

---

9. Start by making a table of amounts which can be spent on exactly two cards:

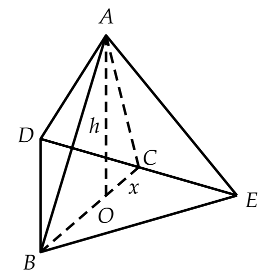
	30	32	36	38	40	62
30	–	62	66	68	70	92
32	62	–	68	70	72	94
36	66	68	–	74	76	98
38	68	70	74	–	78	100
40	70	72	76	78	–	102
62	92	94	98	100	102	–

Since Marni has exactly one of each card, the total amount must be

$$3 \times (\text{one of the amounts in the table}) \leq 238$$

which is the sum of the denominations. The first buyer could not have purchased the \$62 gift card, since $3 \times (30 + 62) = 288 > 238$. The second buyer must have purchased the \$62 card, since $38 + 40 < 2 \times (30 + 32)$, and must have bought 3 or 4 cards, since $62 + 40 < 2 \times (30 + 32)$. The only possibility is $2 \times (30 + 36) = 32 + 38 + 62$ for a total of \$198.

Answer is (A).



10. By symmetry the point O in the face BDE that is nearest to the vertex A must be on the median BC of triangle BDE, and the line segment $\overline{AO}$ must be perpendicular to the face DBE. Let h be the length of the line segment $\overline{AO}$ and x be the length of the line segment $\overline{CO}$, where C is the midpoint of the line segment $\overline{DE}$. The four faces of the tetrahedron are equilateral triangles with side length 2, so the heights of triangles BDE and ADE are, respectively,

$$\left|\overline{BC}\right| = \left|\overline{AC}\right| = \sqrt{3}$$

Then applying Pythagoras' theorem to the triangles ABO and ACO gives

$$\left|\overline{AB}\right|^2 = \left|\overline{BO}\right|^2 + \left|\overline{AO}\right|^2 \Rightarrow 4 = \left(\sqrt{3} - x\right)^2 + h^2$$
$$\left|\overline{AC}\right|^2 = \left|\overline{CO}\right|^2 + \left|\overline{AO}\right|^2 \Rightarrow 3 = x^2 + h^2$$

Solving each of these equations for h^2 gives

$$h^2 = 4 - \left(3 - 2x\sqrt{3} + x^2\right) = 1 + 2x\sqrt{3} - x^2 = 3 - x^2 \Rightarrow 2x\sqrt{3} = 2 \Rightarrow x = \frac{1}{\sqrt{3}}$$

Plugging this value for x into either of the equations for h^2 gives

$$h^2 = 3 - \left(\frac{1}{\sqrt{3}}\right)^2 = 3 - \frac{1}{3} = \frac{8}{3} \Rightarrow h = \frac{2\sqrt{2}}{\sqrt{3}}$$

Answer is (C).

Junior Final, Part B

1. First observe that one of E or N is 0 and the other is 5 in order to produce the digits Y and T in the ones and tens columns. Further, if N is 5 there is a carry to the tens column so that the sum of that column will not give the digit T. Hence, N is 0 and E is 5. Next, observe that since $F \neq S$ there must be a carry from the thousands column to the ten thousands column. The only possible carry is 1, so that $S = F + 1$. Next, observe there must be a carry from the thousands column and the maximum carry in a sum of three digits is 2, the letter I must be either 0 or 1. Since $N = 0$, this means that $I = 1$ and $O = 9$ with a carry of 2 from the hundreds column. Finally, from the observations above applied to the hundreds it is clear that

$$2T + R + 1 = 20 + X \Rightarrow 2T + R - X = 19$$

Eliminating all of the possibilities based on the values already assigned to letters, the possible combinations for F and S are

$$[F, S] = [2, 3], [3, 4], [6, 7], [7, 8]$$

and the possible combinations for T, R, and X are

$$[T, R, X] = [7, 8, 3], [8, 6, 3], [8, 7, 4]$$

The combinations $[T, R, X] = [7, 8, 3]$ and $[T, R, X] = [8, 6, 3]$ are inconsistent with the combinations for F and S. Hence, the only possibility is $[T, R, X] = [8, 7, 4]$. Therefore, $T = 8$. Note that the digit assignments for all of the letters are:

$$E = 5, F = 2, I = 1, N = 0, O = 9, R = 7, S = 3, T = 8, X = 4, Y = 6$$

Answer: The digit represented by the letter T is 8.

2. Applying the definition gives

$$(a \star b) \star c = \left(\frac{a+2b}{2}\right) \star c = \frac{\left(\frac{a+2b}{2}\right) + 2c}{2} = \frac{a+2b+4c}{4}$$

$$a \star (b \star c) = a \star \left(\frac{b+2c}{2}\right) = \frac{a + 2\left(\frac{b+2c}{2}\right)}{2} = \frac{a+b+2c}{2}$$

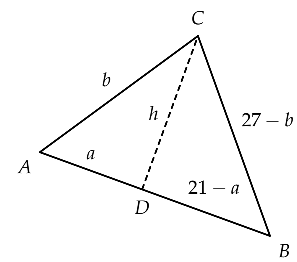
Therefore

$$(a \star b) \star c - a \star (b \star c) = \frac{a+2b+4c}{4} - \frac{a+b+2c}{2} = \frac{a+2b+4c-2a-2b-4c}{4} = -\frac{a}{4}$$

Answer: $-\dfrac{a}{4}$

---

3. Let AB be the side with length 21. If CD is perpendicular to AB, then $h = \overline{CD}$ is the height of the triangle with AB as base. Let $a = \overline{AD}$ and b be the length of the shortest side of the triangle. (See the diagram.) If b is the shortest side of the triangle, then $b \le 13$. Using Pythagoras' with triangles ACD and BCD gives



$$b^2 = a^2 + h^2 \quad \text{and} \quad (21-a)^2 + h^2 = (27-b)^2$$

Using the fact that $b^2 = a^2 + h^2$, the second equation can be written as

$$21^2 - 42a + a^2 + h^2 = 27^2 - 54b + b^2 \Rightarrow 54b - 42a = 27^2 - 21^2 = (27-21)(27+21)$$

Then factoring and dividing by 6 gives

$$9b - 7a = 48 \Rightarrow 7a = 9b - 48$$

In order for a to be positive, it must be true that $b \ge 6$. Hence, $6 \le b \le 13$. The only value of b in this range making a an integer is $b = 10$. In this case, $a = 6$ and $\overline{CB} = 17$. Therefore, the shortest side of the triangle is 10.

Alternative solution:

Using Heron's formula gives:

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}, \qquad s = \frac{a+b+c}{2}$$

Here $s = 24$, $b = 21$, $c = 27 - a$ so

$$\text{Area} = \sqrt{24(24-a)3(a-3)} = 6\sqrt{2(24-a)(a-3)}$$

Substituting $a = 4, 5, 6, \ldots, 13$ gives an integral area only when $a = 10$.

Answer: The shortest side of the triangle is 10.

4. Start with the inequality on the left, and observe that $1 < \sqrt{2}$,

$$\frac{1}{\sqrt{2}-1} = \left(\frac{1}{\sqrt{2}-1}\right)\left(\frac{\sqrt{2}+1}{\sqrt{2}+1}\right) = \frac{\sqrt{2}+1}{2-1}$$
$$= \sqrt{2}+1 < \sqrt{2}+\sqrt{2} = 2\sqrt{2}$$

Now for the inequality on the right, observe that $\sqrt{3} > \sqrt{2}$, so that

$$\frac{1}{\sqrt{3}-\sqrt{2}} = \left(\frac{1}{\sqrt{3}-\sqrt{2}}\right)\left(\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}}\right) = \frac{\sqrt{3}+\sqrt{2}}{3-2}$$
$$= \sqrt{3}+\sqrt{2} > \sqrt{2}+\sqrt{2} = 2\sqrt{2}$$

Combining the two inequalities gives the required result.

Answer: See proof above.

---

5. Let x denote the number of dollars, and let y denote the number of cents in the original cheque. The variables x and y represent positive integers with no more than two digits, that is, $0 < x, y < 100$. Then the value, in cents, of the original cheque is $C = 100x + y$, and the number of cents received by Ms Smith is $A = 100y + x$. The number of cents that Ms Smith has after buying the newspaper is

$$A - 50 = 3C \Rightarrow (100y + x) - 50 = 3\,(100x + y) \Rightarrow 97y - 299x = 50$$

Note that $299 = 3\,(97) + 8$ and write the equation above as

$$97y = 3\,(97)\,x + 8x + 50$$

In order for y to be an integer, the expression $8x + 50$ must be a multiple of 97. This gives

$$8x + 50 = 97n$$

where n is a positive integer, otherwise x would be negative. Note further, that for x to be an integer, n must be a multiple of 2. Starting with $n = 2$ gives

$$8x + 50 = 97\,(2) = 194 \Rightarrow 8x = 144 \Rightarrow x = 18$$

The next value of n for which x is an integer is $n = 2 + 8 = 10$. This gives $8x = 920 \Rightarrow x = 115 > 100$. Hence, only $n = 2$ gives a valid solution. For $x = 18$, the value of y is

$$97y = 3\,(97)\,(18) + (97)\,(2) \Rightarrow y = 3\,(18) + 2 = 56$$

Hence, the original amount of the cheque was \$18.56.

Problem 5 continued ...

...Problem 5 continued

Alternative solution:

Starting with the equation $97y - 299x = 50$, since 97 and 299 are both prime, their greatest common factor is 1. This means that there exist integers s and t such that $97s - 299t = 1$. Using the facts that

$$299 = 3\,(97) + 8 \ \text{ and } \ 97 = 12\,(8) + 1$$

solving the first equation for 8 gives

$$8 = 299 - 3\,(97)$$

Then substituting into the second equation gives

$$97 = 12\,[299 - 3\,(97)] + 1 \Rightarrow 97 + 36\,(97) - 12\,(299) = (97)\,(37) - (299)\,(12) = 1$$

Multiplying this equation by 50 gives

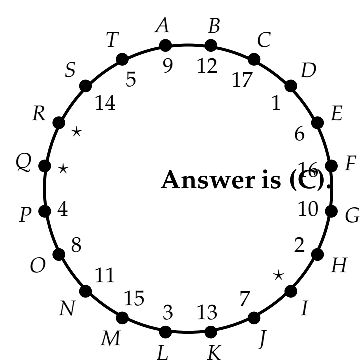
$$97\,(37)\,(50) - 299\,(12)\,(50) = 50 \Rightarrow 97\,(1850) - 299\,(600) = 50$$

It follows that one possible solution to the equation is $y = 1850$ and $x = 600$. Any other solution to the equation will give $y = 1850 - 299k$ and $x = 600 - 97k$, where k is an integer. Taking $k = 6$ gives $y = 1850 - 1794 = 56$ and $x = 600 - 582 = 18$. Giving the same answer as above. Note that any other value of k gives values of x and y outside of the range required by the problem.

Answer: The original amount of the cheque is \$18.56.

Senior Final, Part A

1. The order in which the balloons are popped is shown by the numbering on the diagram. The last three unpopped balloons are indicated with a $\star$. From the diagram, unpopped balloons are I, Q, and R.



Answer is (C).

---

2. The maximum number of points in a 13-card hand corresponds to four Aces, four Kings, four Queens, and one Jack for a total of $4 \times 4 + 4 \times 3 + 4 \times 2 + 1 = 16 + 12 + 8 + 1 = 37$.

Answer is (D).

---

3. Write the fraction as

$$\frac{87}{17} = 5 + \frac{2}{17} = 5 + \frac{1}{\left(\frac{17}{2}\right)} = 5 + \frac{1}{8 + \frac{1}{2}}$$

This gives $w = 5$, $y = 8$, and $x = 2$, for a sum of $x + y + z = 15$.

Problem 3 continued ...

. . . Problem 3 continued

Alternative solution:

Simplifying the fraction on the right gives

$$w + \frac{1}{y + \frac{1}{x}} = w + \frac{1}{\left(\frac{xy+1}{x}\right)} = w + \frac{x}{xy+1} = \frac{wxy + w + x}{xy+1} = \frac{87}{17}$$

Since 87 and 17 are relatively prime, so there is no cancelation in the fraction, matching the denominators gives

$$xy + 1 = 17 \implies xy = 16$$

Then matching the numerators and using $xy = 16$ gives

$$xyw + w + x = 16w + w + x = 17w + x = 87 \implies 17w = 87 - x$$

For w to be an integer the righthand side $87 - x$ must be a multiple of 17. The only positive integer values of x that make $87 - x$ a positive multiple of 17 are 70, 53, 36, 19, and 2. Only $x = 2$ makes y an integer. For $x = 2$, $17w = 85 \implies w = 5$. Therefore, the values of x, y, and w are $x = 2$, $y = 8$, and $w = 5$, so that $x + y + w = 15$.

Answer is (E).

---

4. Consider the table:

p	2^{p-1}	$\frac{2^{p-1}-1}{p}$	square?
3	4	1	yes
5	16	3	no
7	64	9	yes
11	1024	93	no
13	4096	315	no

Two values of p give a perfect square, $p = 3$ and $p = 7$. The sum of the two values is $3 + 7 = 10$.

Answer is (C).

---

5. See Junior Final Part A, Problem 5.

Answer is (D).

---

6. The given equation can be written as

$$x = 2\sqrt{4 + 2\sqrt{4 + 2\sqrt{4 + 2\sqrt{4 + \cdots}}}} = 2\sqrt{4 + x} \implies x^2 = 16 + 4x \implies x^2 - 4x - 16 = 0$$

Solving the quadratic equation for x, taking the positive root, gives $x = \dfrac{4 + \sqrt{16 + 4 \times 16}}{2} = 2\left(1 + \sqrt{5}\right)$.

Answer is (A).

7. Note that $100 = 2(16) + 4(17)$, so a score of exactly 100 can be achieved with no more than 6 darts. Further calculation (based partly on estimation) reveals that no sum with fewer than 6 darts is possible.

Answer is (D).

---

8. See Junior Final A, Problem 10 for the notation and values for the appropriate lengths. Let P be the point on AO that is equidistant from the four vertices of the tetrahedron. This will be the centre of the sphere and the radius will be the common length

$$r = |\overline{AP}| = |\overline{BP}| = |\overline{DP}| = |\overline{EP}|$$

Applying Pythagoras' Theorem to triangle BPO gives

$$\begin{aligned} r^2 &= |\overline{PO}|^2 + |\overline{BO}|^2 = (h-r)^2 + \left(|\overline{BC}| - x\right)^2 \\ &= \left(\frac{2\sqrt{2}}{\sqrt{3}} - r\right)^2 + \left(\sqrt{3} - \frac{1}{\sqrt{3}}\right)^2 = \left(\frac{2\sqrt{2}}{\sqrt{3}} - r\right)^2 + \left(\frac{2}{\sqrt{3}}\right)^2 \\ &= \frac{8}{3} - \frac{4\sqrt{2}}{\sqrt{3}}r + r^2 + \frac{4}{3} \end{aligned}$$

Canceling the r^2 from both sides and solving for r gives

$$\frac{4\sqrt{2}}{\sqrt{3}}r = \frac{12}{3} = 4 \Longrightarrow r = \frac{\sqrt{3}}{\sqrt{2}}$$

Answer is (C).

---

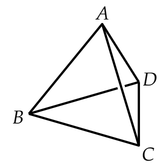
9. Let v_1 be the speed of the slower train and v_2 be the speed of the faster one. When they move in opposite directions their relative speed is $v_1 + v_2$ and they take 5 seconds to move 180 metres. If they move in the same direction then their relative speed is $v_2 - v_1$ and they take 15 seconds to move 180 metres. This gives

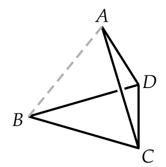
$$\left.\begin{aligned} 5\,(v_1 + v_2) &= 180 \\ 15\,(v_2 - v_1) &= 180 \end{aligned}\right\} \Longrightarrow \left.\begin{aligned} v_1 + v_2 &= 36 \\ -v_1 + v_2 &= 12 \end{aligned}\right\}$$

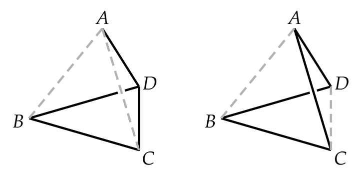
Subtracting the two equations gives $2v_1 = 24 \Longrightarrow v_1 = 12$. Therefore, the speed of the slower train is 12 m/s.

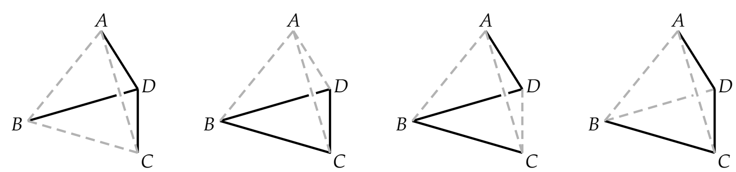
Answer is (B).

10. Suppose the colours are red (shown as dashed gray in the diagram) and black. The possibilities for the colourings are shown the following diagrams.

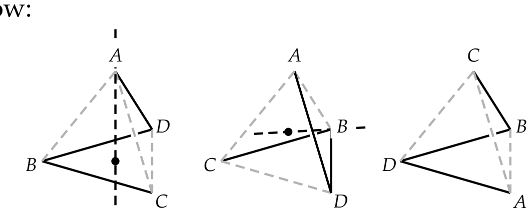
- Six black, no red edges: 

- Five black, one red edge: 

- Four black, two red edges: 

- Three black, three red edges: 

For the first three cases the two colours can be interchanged, so each is counted twice. In the fourth case, the number of black and red sides is the same, so they are counted only once. Further, for any other possible colouring the tetrahedron can be oriented to produce one of the colourings shown above, using rotations about an axis through one of the vertices and the centroid of the opposite face. Hence, for example, any colouring with only one red edge is indistinguishable from the second colouring shown above. To verify that the last two colourings in the fourth case above are distinguishable, observe that the only way to reorient the tetrahedron in the first of the two cases so that it could look the same as the second case is to have vertex C in the place of vertex A, vertex D in the place of vertex B, and vertex A in the place of vertex C. This can be achieved by first rotating about the axis through vertex A and the centroid of the opposite face, counterclockwise (as viewed from the vertex A toward the centroid) through a 120° angle and then rotating about the axis through the new vertex B and the the centroid of the opposite face, clockwise (as viewed from the vertex B toward the centroid) through a 120° angle, as shown below:



The resulting colouring is identical to the original colouring.
Therefore, the total number of ways to paint the edges is: $1 + 1 + 2 + 4 + 2 + 1 + 1 = 12$.

Answer is (A).

Senior Final, Part B

1. In order for n to appear as the last three digits of n^2 the value of $n^2 - n = n\,(n-1)$ must be a multiple of 1000. Noting that $1000 = 2^3 5^3$, we see that n or $n-1$ must be a multiple of 125 and the other a multiple of 8. Thus, $n = 8p$ and $n - 1 = 125q$, or $n = 125p$ and $n - 1 = 8p$, where p and q are positive integers. Therefore

$$8p = 125q + 1 \ \text{ or } \ 8p = 125q - 1$$

When $q = 3$, $125q + 1 = 376$, which is a multiple of 8, giving $n = 376$. When $q = 5$, $125q - 1 = 624$, which is a multiple of 8, giving $n = 625$. These are the only possibilities for which one of $125q + 1$ or $125q - 1$ is a multiple of 8 and n is a three digit number. Therefore, the only two three digit numbers for which n appears as the last three digits of n^2 are $n = 376$ and $n = 625$.

Answer: The numbers are 376 and 625.

---

2. Start with the inequality on the left

$$\begin{aligned}\sqrt{n+1} - \sqrt{n} &= \left(\sqrt{n+1} - \sqrt{n}\right)\left(\frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}}\right) = \frac{(n+1) - n}{\sqrt{n+1} + \sqrt{n}} \\ &< \frac{1}{\sqrt{n} + \sqrt{n}} = \frac{1}{2\sqrt{n}}\end{aligned}$$

since $\sqrt{n+1} > \sqrt{n}$. Now for the inequality on the right

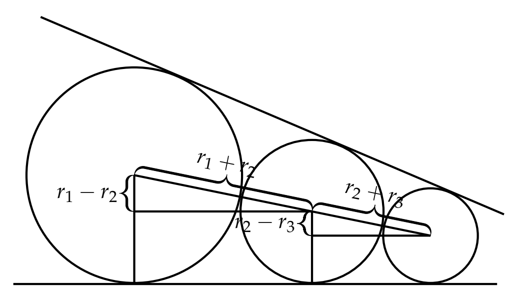
$$\begin{aligned}\sqrt{n} - \sqrt{n-1} &= \left(\sqrt{n} - \sqrt{n-1}\right)\left(\frac{\sqrt{n} + \sqrt{n-1}}{\sqrt{n} + \sqrt{n-1}}\right) = \frac{n - (n-1)}{\sqrt{n} + \sqrt{n-1}} \\ &> \frac{1}{\sqrt{n} + \sqrt{n}} = \frac{1}{2\sqrt{n}}\end{aligned}$$

since $\sqrt{n-1} < \sqrt{n}$. Combining the two inequalities gives the required result.

Answer: See the proof above.

---

3. First consider three marbles with radii r_1, r_2, and r_3 for which $r_1 > r_2 > r_3$. Suppose that the radii are such that when the marbles are placed on a horizontal surface tangent to each other, a plane tangent to the largest and smallest marble is also tangent to the middle one. See the diagram. Then, by similar triangles



$$\begin{aligned}\frac{r_1 + r_2}{r_1 - r_2} &= \frac{r_2 + r_3}{r_2 - r_3} \\ \Rightarrow r_1 r_2 - r_1 r_3 + r_2^2 - r_2 r_3 &= r_1 r_2 + r_1 r_3 - r_2^2 - r_2 r_3 \\ \Rightarrow 2r_2^2 = 2r_1 r_3 &\Rightarrow \frac{r_1}{r_2} = \frac{r_2}{r_3}\end{aligned}$$

Now consider five marbles with radii $18 < r_1 < r_2 < r_3 < 8$. The radius of the middle marble is r_2. Using the result above gives

$$\frac{18}{r_1} = \frac{r_1}{r_2} = \frac{r_2}{r_3} = \frac{r_3}{8}$$

Problem 3 continued ...

...Problem 3 continued

The far left and far right ratios give

$$18\,(8) = 144 = r_1 r_3$$

and from the middle two ratios

$$r_1 r_3 = r_2^2$$

Hence

$$r_2^2 = 144 \Rightarrow r_2 = 12$$

Therefore, the radius of the middle marble is $r_2 = 12$.

---

4. Let B represent "Mr Baker," b represent "baker," etc. Construct a table showing the possible occupations of the each of the people, with columns labeled with each person's name and the rows labeled with each occupation. Each entry in the table will have a

- × indicating that the name and occupation combination represented by the entry is impossible
- a number, or its negation, such as 1 or ~1, indicating that Statement 1, or its negation, makes the name and occupation combination represented by the entry is impossible,
- a number, or its negation, in a circle, such as (1) or (~1), indicating that Statement 1, or its negation, gives the occupation for the name represented by the entry
- just a circle, indicating that the entry gives the occupation for the name repreented by the entry, by elimination from other entries

If all of the statements are true, then the table looks like this

	B	C	D	P
b	1	2	(2)	2
c	1	◯	2	(4)
d	1	×	2	◯
p	(1)	1	1	1

where more than one statement determines the value of a given entry, the first numerically is used. Note that Statement 3 is redundant, since Statement 1 means that Mr Baker is the plumber, so it is already true that Ms Carpenter is not the plumber. From the table above it is clear that Ms Carpenter must be the carpenter. This contradicts that statement that none of the four people have a name that identifies their occupation. Note that this requirement means that there is a × along the diagonal of the table.

First note that if Statement 1 is the only statement that is true, then by Statement 1 Mr Baker is the plumber, but, if Statement 3 is false, then Ms Carpenter must be the plumber. This is a contradiction, so Statement 1 must be one of the false statements.

Problem 4 continued ...

...Problem 4 continued

If Statement 2 is the only true statement, the table is:

	B	C	D	P
b	×	2	(2)	2
c	~4	×	2	(~4)
d	○	~3	×	~4
p	~1	(~3)	2	×

In this case, Mr Baker is the driver.
If Statement 3 is the only true statement, the table is:

	B	C	D	P
b	×	○	~2	~4
c	~4	×	~4	(~4)
d	○	×	×	~4
p	~1	~3	○	×

Again in this case, Mr Baker is the driver.
If Statement 4 is the only true statement, the table is:

	B	C	D	P
b	×	~3	~2	○
c	×	×	○	×
d	○	~3	×	×
p	~1	(~3)	~3	×

Again in this case, Mr Baker is the driver. Hence, the only possibility if three of the four statements are false is that Mr Baker is the driver.

Answer: Mr. Baker is the driver.

5. Let x be the distance from upper right corner to the fold point along the upper edge. After the fold, this length goes along the upper edge of the fold from the fold point to the midline, as shown in the diagram. From the small right triangle we have

$$x^2 = (1-x)^2 + \left(2-\sqrt{3}\right)^2 \Rightarrow x = \frac{1+\left(2-\sqrt{3}\right)^2}{2}$$
$$\Rightarrow 1-x = \frac{1-\left(2-\sqrt{3}\right)^2}{2} = \frac{4\sqrt{3}-6}{2} = \sqrt{3}\left(2-\sqrt{3}\right)$$

The area of the small triangle is

$$\left(2-\sqrt{3}\right)(1-x) = \sqrt{3}\left(2-\sqrt{3}\right)^2 = \sqrt{3}\left(7-4\sqrt{3}\right)$$

while the area of the large triangle is $\sqrt{3}$. Hence, ratio of the areas is $7-4\sqrt{3} : 1$.