BRITISH COLUMBIA SECONDARY SCHOOL MATHEMATICS CONTEST, 2010

Solutions

Junior Preliminary

1. Writing each fraction over the lowest common denominator of 60 gives, in the order above

$$\frac{20}{60}, \frac{24}{60}, \frac{22}{60}, \frac{25}{60}, \frac{21}{60}$$

Clearly, the largest is $\frac{5}{12} = \frac{25}{60}$

Answer is (D).

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2. The time in question is the least time t such that $3.5t$ and $4t$ are both integers. Evidently, this occurs when $t = 2$.

Alternative solution:
Let d_A and d_B be the distances that Antonino and Boris run in t hours. Then $d_A = 3.5t$ and $d_B = 4t$. They are at the same point at the same time when $d_B = d_A + n$ where n is any positive integer. This gives $4t = 3.5t + n \Rightarrow t = 2n$. For $n = 1$, $t = 2$ and both runners are at the starting point at the same time. This is obviously the earliest time that they will next be at their starting point at the same time.

Answer is (D).

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3. Suppose, in the first place, that the cards have kinds (i.e., types), but no suits attached. Place a Queen to the right of some King and place another Queen to the left of the first Queen. Now label the cards spade, spade, and heart to arrive at the configuration

$$\spadesuit\textbf{King} \quad \spadesuit\textbf{Queen} \quad \heartsuit\textbf{Queen}$$

Since this configuration satisfies the conditions of the problem, and it is evident no smaller configuration will work, it follows that the minimum number of cards in the row is 3.

Answer is (B).

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4. Pages 1–9 take 9 digits. Pages 10–99 (90 pages at 2 digits each) take 180 digits for a running total of 189 digits. Pages 100–999 take three digits each; we have $255 - 189 = 66$ digits with which we can number 22 pages. Consequently there are $99 + 22 = 121$ pages.

Answer is (B).

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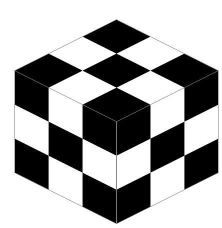
5. There is only one way to arrange the cubes, as shown in the figure. There are 5 black cubes on the top and bottom layers and 4 in the middle layer for a total of 14 black cubes.



Answer is (C).

6. There are 9 small triangles with adjacent cevians, there are 8 triangles made up of pairs of adjacent small triangles, and so on, for a total of

$$9+8+7+6+5+4+3+2+1=45$$

triangles.

Alternative solution:

Triangles are formed by choosing two of the 10 lines from the top vertex. This can be done in

$${}_2C^{10}=\frac{10\times 9}{2}=45$$

ways.

Answer is (D).

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7. The new number is given by

$$2\times\left(\frac{5248}{2}-213+312\right)=2\times(2624+99)=5446$$

Alternative solution:

Let the numbers in the set be x_1, x_2, x_3, ..., x_n and the sum of these numbers be S. Without loss of generality let the changed number be x_1, then

$$S=x_1+x_2+x_3+...+x_n=\frac{5248}{2}=2624$$

so that

$$312+x_2+x_3+...+x_n=99+213+x_2+x_3+...+x_n=99+2624=2723$$

Hence twice, the sum of the new set of numbers is 5446.

Answer is (E).

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8. First note that if there are no quarters, since there are three times as many one dollar coins as two coins, there must be 36 one dollar coins and 12 two dollar coins. This gives a total of 60 dollars. Next note that the number of quarters must be a multiple of four in order for there to be a whole number of dollars in the jar. Further, for every 4 quarters that are added, which amounts to 1 dollar, a combination of 4 one dollar and two dollar coins must be removed. Again, since there three times as many one dollar coins as two dollar coins, this means that three one dollar coins and one two dollar coin are removed, which amounts to 5 dollars. Hence, for every 4 quarters added there is a net decrease of $5-1=4$ dollars in the jar. Therefore, with 4 quarters in the jar, there will 56 dollars and with 8 quarters there will be 52 dollars, as required. So, there must be 8 quarters in the jar.

Problem 8 continued . . .

...Problem 8 continued

Alternative solution:

If q, ℓ, and t, denote, respectively, the number of quarters, loonies (one dollar coins), and toonies (two dollar coins), then these variables satisfy the linear system

$$\begin{aligned} 25q + 100\ell + 200t &= 5200 \\ q + \ell + t &= 48 \\ \ell &= 3t \end{aligned}$$

Dividing the first equation by 25 and substituting $\ell = 3t$ into the other two equations gives

$$\begin{aligned} q + 20t &= 208 \\ q + 4t &= 48 \end{aligned}$$

Subtracting the two equations gives $16t = 160 \Rightarrow t = 10$. Using either of the two equations then gives $q = 8$.

Answer is (A).

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9. Drop a perpendicular from E to the side DC and label the base K. Since $\overline{EK} = \overline{BC} = 9$, by Pythagoras' theorem, $\overline{DK} = \overline{AE} = 12$, and hence $\overline{EB} = 3$. Since $\angle DEF = 90°$, we see

$$\angle AED + \angle BEH = 90° = \angle BEH + \angle EHB$$

so $\angle EDK = \angle EHB$ and $\angle BEH = \angle KED$. It follows that triangles DEK and HEB are similar so $\overline{BH} = 4$ and $\overline{HC} = 5$. Finally, we compute the area of the quadrilateral $DEHC$ as the sum of the areas of the triangle DEK and the trapezoid $KEHC$, giving

$$\text{Area } DEHC = \frac{1}{2} \cdot 12 \cdot 9 + \left(\frac{5+9}{2}\right) \cdot 3 = 54 + 21 = 75$$

Answer is (D).

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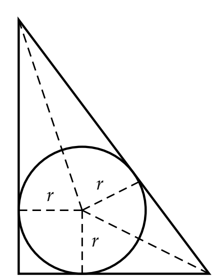
10. The circle so-defined is called the *incircle* of the given triangle. Drop perpendiculars of length r from the centre of the incircle to each side of the given triangle and connect the centre to each vertex of the triangle. (See the diagram.) Equating the areas of the large triangle with the sum of the areas of the three triangles just constructed, we find

$$\tfrac{1}{2}(5r) + \tfrac{1}{2}(3r) + \tfrac{1}{2}(4r) = \tfrac{1}{2}(4)(3) \Rightarrow 6r = 6 \Rightarrow r = 1$$

so the radius is 1.



Answer is (A).

11. The expression $\dfrac{12}{n-3}$ is an integer, exactly when $n-3$ is a factor of 12. Since $12 = 2^2 \cdot 3^1$, there are $(2+1)(1+1) = 6$ positive integer factors, and hence 12 integer factors. Hence,

$$n - 3 = -12,\ -6,\ -4,\ -3,\ -2, -1,\ 1,\ 2,\ 3,\ 4,\ 6,\ 12$$

The values -12, -6, and -4 correspond to negative values of n. So there are $12 - 3 = 9$ values n satisfying the conditions of the problem.

Answer is (C).

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12. Since the sum of the numbers in the set is 36, the minimum number is 1, and the maximum number is 8, any subset of the type described in the problem has minimum sum $36 - (8+7) = 21$, Likewise, the maximum sum of any such subset is $36 - (1+2) = 33$. The only two digit number whose digits add to ten and that lies in the indicated range is 28. Hence, the sum of the numbers removed must be 8.

Answer is (E).

Senior Preliminary

1. Convert to rational form. The base is increased by one tenth, or to $\frac{11}{10}$ of its original value. The height is decreased by one twentieth, or to $\frac{19}{20}$ of its original value. Since the area of a triangle varies directly as the product of the base and the height, the area of the new triangle is the area of the old one multiplied by the product of the scaling factors $\frac{11}{10}$ and $\frac{19}{20}$. Since

$$\left(\frac{11}{10}\right)\left(\frac{19}{20}\right) = \frac{209}{200} = 1 + \frac{9}{200} = 1 + 0.045$$

the area is increased by 4.5 percent.

Alternative solution:

If the original triangle has base b and height h, then the area of the new triangle is

$$(1.1h)(0.95h) = 1.045bh$$

Hence, the area is increased by 4.5%.

Answer is (D).

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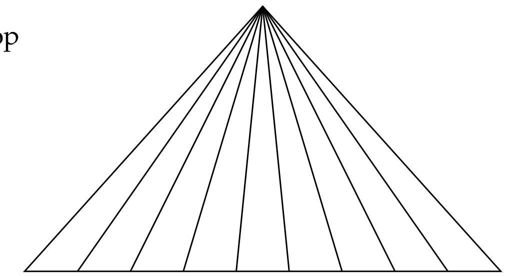
2. If Alan enters a vertex by one road he must leave it by a different one (unless it is the end of his trip). With the possible exception of where he begins and ends, all the vertices must have an even number of roads. Counting the roads at each vertex, we see that A has 2, B and D have 3 and the others have 4. Therefore, he must start (or end) at either B or D.

Answer is (B).



3. We get triangles by choosing two of the 10 lines from the top vertex. This can be done in

$$ {}_2C^{10} = \frac{10 \times 9}{2} = 45 $$

ways.

Answer is (D).

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4. Since

$$ 30^2 + 40^2 = 14^2 + 48^2 = 50^2 $$

it is clear that $\overline{AC} = 50$, and that both triangles ABC and ADC are right triangles by Pythagoras' theorem. It follows from the standard formula for the area of a triangle, that

$$ \text{Area of } ABCD = \frac{1}{2}\,(30)\,(40) + \frac{1}{2}\,(14)\,(48) = 936 $$

Alternative solution:

This result follows more directly from Brahmagupta's formula for the area A of a cyclic quadrilateral, a quadrilateral with all of its vertices on a circle, with sides a, b, c, and d, and *semi-perimeter*

$$ s = \frac{a+b+c+d}{2} $$

as

$$ A = \sqrt{(s-a)(s-b)(s-c)(s-d)} $$

Here $a = 30$, $b = 40$, $c = 14$, $d = 48$, and $s = 66$, so that the area is

$$ \begin{aligned} A &= \sqrt{(66-30)\,(66-40)\,(66-14)\,(66-48)} \\ &= \sqrt{(36)\,(26)\,(52)\,(18)} = \sqrt{(2^2)\,(18^2)\,(26^2)} = 936 \end{aligned} $$

Answer is (A).

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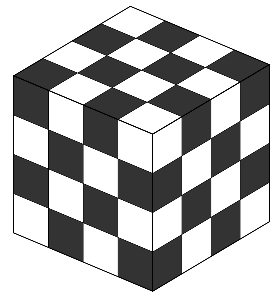
5. The possible total sums taking four choices at a time are: 34, 35, and 38. The only one of these that lies among the choices is 34.

Answer is (C).

6. A possible arrangement of the cubes that meets the specifications in the problem is shown in the figure. Any other arrangement meeting the specifications can be obtained from this one through rotation. There are 8 black cubes on the top face and 8 on the bottom face, 4 black cubes on the front and 4 on the back face that are not also on the the top or bottom face, and 2 black cubes on the left face and 2 on the right face that have not already been counted. This gives a total of $8 + 8 + 4 + 4 + 2 + 2 = 28$ visible black cubes. The visible faces of the cube surround an inner $2 \times 2 \times 2$ cube which contains 8 smaller cubes. If all of 8 of these smaller cubes are black, which meets the specifications since none of them are visible and gives the maximum possible number of black cubes, there is total of $28 + 8 = 36$ black cubes.



Alternative solution:

Count the visible black faces using the inclusion-exclusion principle. There are 8 black cubes on each of the six faces of the large cube for a total of $6 \times 8 = 48$ cubes. This counts the cubes on the edges of the large cube twice. There are 2 black cubes on each of the twelve edges of the large cube for a total of $12 \times 2 = 24$ cubes. This counts cubes at the corners of the large cube twice. There 4 black cubes on the corners of the large cube. So the total number of visible black cubes is $48 - 24 + 4 = 28$, as above. From here the solution is the same as given above.

Answer is (C).

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7. The triangle CBE formed by the points B and C, and the intersection point E of the two semicircles is equilateral since its sides all have length equal to the radius of the semicircles. Hence, the measure of the central angle subtended by the arc of either semicircle is 60°. Therefore, using the inclusion-exclusion principle, the area of the shaded region is twice the area of the circular segment of angle 60° minus the area of the equilateral triangle, given by

$$\begin{aligned}\text{area of shaded region} &= 2\left[\left(\frac{1}{2}\right)\left(\frac{\pi}{3}\right)\left(2\sqrt{3}\right)^2\right] - \left(\frac{1}{2}\right)\left(2\sqrt{3}\right)(3)\\ &= 4\,(3)\left(\frac{\pi}{3}\right) - 3\sqrt{3} = 4\pi - 3\sqrt{3}\end{aligned}$$

Answer is (B).

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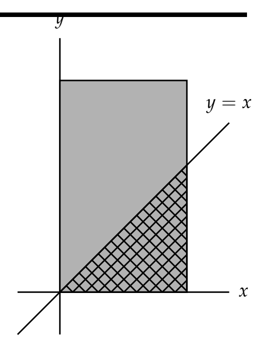
8. The required probability is the ratio of the area of the triangle formed by the line $x = y$, and the positive x-axis, to the area of the rectangle formed by the given points; that is:

$$\frac{\left(\dfrac{9}{2}\right)}{15} = \frac{3}{10}$$



Answer is (A).

9. Obviously the shaded strip has uniform width. Hence, triangles ABD and ACD are congruent 30°-60°-90° triangles. Hence,

$$\overline{AB} = \overline{AC} = \left(\sqrt{3}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{3}{2}$$

The same is true at the other two vertices, so that the side length of the outer triangle is $5 + \frac{3}{2} + \frac{3}{2} = 8$. Therefore, the area of the shaded region between the two triangles is

$$\frac{1}{2}(8)\left[8\left(\frac{\sqrt{3}}{2}\right)\right] - \frac{1}{2}(5)\left[5\left(\frac{\sqrt{3}}{2}\right)\right] = \frac{\sqrt{3}}{4}(64-25) = \frac{39\sqrt{3}}{4}$$

Answer is (C).

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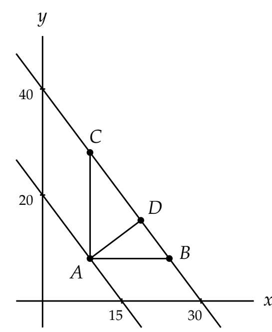
10. **Alternative**

Take a point A on the line $4x + 3y = 60$, and let B and C be the respective intersection points of the horizontal and vertical from A to the other line. From the diagram it is clear that $\overline{AC} = 20$, and $\overline{AB} = 15$. Drop a perpendicular from A to D on the second line. The triangles BAD and BCA are similar and by Pythagoras' theorem $\overline{BC} = 25$. It follows that



$$\frac{\overline{AD}}{20} = \frac{15}{25} \Rightarrow \overline{AD} = 12$$

Solution:

The line $3x - 4y = 45$ is perpendicular to the two given lines and goes through the point $(15, 0)$ on the line $4x + 3y = 60$. To find the x-coordinate of the point of intersection of this line with the line $4x + 3y = 120$, first solve for y in the equation of the perpendicular line:

$$y = \frac{3}{4}x - \frac{45}{4}$$

Then substitute into the other line to give

$$4x + 3\left(\frac{3}{4}x - \frac{45}{4}\right) = 4x + \frac{9}{4}x - \frac{135}{4} = \frac{25}{4}x - \frac{135}{4} = 120$$
$$\Rightarrow \frac{25}{4}x = \frac{615}{4} \Rightarrow x = \frac{615}{25} = \frac{123}{5}$$

The y-coordinate of the point of intersection is

$$y = \frac{3}{4}\left(\frac{123}{5}\right) - \frac{45}{4} = \frac{369-225}{20} = \frac{144}{20} = \frac{36}{5}$$

Hence, the point of intersection has coordinates $\left(\frac{123}{5}, \frac{36}{5}\right)$. The perpendicular distance between two given parallel lines is the distance between this point of intersection and the point $(15, 0)$. This distance is

$$\sqrt{\left(\frac{123}{5} - 15\right)^2 + \left(\frac{36}{5}\right)^2} = \sqrt{\left(\frac{48}{5}\right)^2 + \left(\frac{36}{5}\right)^2} = \frac{12}{5}\sqrt{4^2+3^2} = 12$$

Answer is (E).

11. Let R be the radius of the ball. Putting a circle of radius R into the hole gives a chord of length 48 cm. The midpoint of the chord is 6 cm from the bottom of the hole, or $R - 6$ cm from the centre of the circle. This gives a right triangle with hypotenuse R and perpendicular sides of length 24 and $R - 6$. Using Pythagoras' theorem gives

$$R^2 = 24^2 + (R-6)^2 = 24^2 + R^2 - 12R + 6^2 \Rightarrow 12R = 24^2 + 6^2 = 12\,(48+3) \Rightarrow R = 51$$

Answer is (D).

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12. In order to obtain the maximum possible number of pieces, each cut line must intersect all of the previous cut lines and no more than two lines can intersect at the same point. Assuming that these conditions are satisfied when the n^{th} cut is made it adds an additional n pieces. Let P_n be the number of pieces when there are n cuts. Then the statement above shows that

$$P_n = P_{n-1} + n$$

Further, when there are no cuts, there is one piece of pizza. So that $P_0 = 1$. Hence,

$$\begin{aligned} P_1 &= 1+1 \\ P_2 &= 1+1+2 \\ P_3 &= 1+1+2+3 \\ P_4 &= 1+1+2+3+4 \end{aligned}$$

From this it is clear that the number of pieces of pizza when there are n cuts is

$$P_n = 1 + 1 + 2 + \cdot + n = 1 + \frac{n\,(n+1)}{2}$$

Therefore, when there are 100 cuts, the number of pieces of pizza is

$$P_{100} = 1 + \frac{100 \times 101}{2} = 1 + 50 \times 101 = 5051$$

Answer is (E).

Junior Final, Part A

1. The first six numbers, all of which have 1 as the left-most digit, are: 1234, 1243, 1324, 1342, 1423, 1432. The next six have 2 as the left-most digit. The numbers with 3 as the left-most digit are: 3124, 3142, 3214, 3241, 3412, 3421. The 14^{th} number is the second of the numbers with 3 as the left-most digit, which is 3142.

Answer is (C).

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2. The digits in any three digit number that satisfy the given conditions are: $(3,3,7)$, $(3,4,6)$, $(3,5,5)$, and $(4,4,5)$. Three of these possibilities have two digits that are the same. Each of these can be arranged to form a distinct three digit number in 3 ways, for a total of nine numbers. The remaining possibility has three distinct digits which can be arranged to form a distinct three digit number in 6 ways. Hence, the total number of three digit numbers that satisfy the given conditions is $9 + 6 = 15$.

Answer is (D).

3. The sides of the triangle are $1+2=3$, $1+3=4$, and $2+3=5$ which is a right triangle. Its area is $\frac{1}{2}(3\times 4)=6$.

Answer is (C).

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4. Since there are $3\cdot 2\cdot 1=3!=6$ ways for the women to pick up hats and in only one of them do all three women get their own hats, the required probability is $\frac{1}{6}$.

Answer is (A).

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5. Since $a_1=1$ and $a_{n+1}=1+\dfrac{1}{a_n}$, we see that

$$\begin{array}{llll}
a_2=1+\dfrac{1}{1}=2 & a_3=1+\dfrac{1}{2}=\dfrac{3}{2} & a_4=1+\dfrac{2}{3}=\dfrac{5}{3} & a_5=1+\dfrac{3}{5}=\dfrac{8}{5}\\
a_6=1+\dfrac{5}{8}=\dfrac{13}{8} & a_7=1+\dfrac{8}{13}=\dfrac{21}{13} & a_8=1+\dfrac{13}{21}=\dfrac{34}{21} & a_9=1+\dfrac{21}{34}=\dfrac{55}{34}\\
a_{10}=1+\dfrac{34}{55}=\dfrac{89}{55} & a_{11}=1+\dfrac{55}{89}=\dfrac{144}{89} & a_{12}=1+\dfrac{89}{144}=\dfrac{233}{144} &
\end{array}$$

From the results above it is clear that the first value of $n>2$ for which the denominator is a perfect square is $n=12$.

Note: If

$$a_n=\frac{b_n}{c_n}$$

then

$$a_{n+1}=1+\frac{c_n}{b_n}=\frac{b_n+c_n}{b_n}=\frac{b_{n+1}}{c_{n+1}}$$

which means that $c_{n+1}=b_n$ and $b_{n+1}=b_n+c_n=b_n+b_{n-1}$, so the numerators (and denominators) are the terms in the Fibonacci sequence.

Answer is (B).

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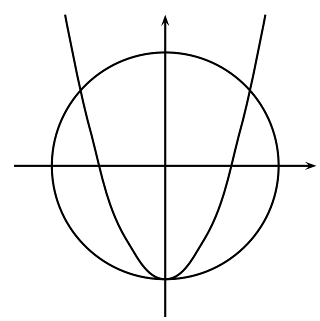
6. The circle $x^2+y^2=9$ has radius 3 and is centred at the origin, and the parabola $y=x^2-3$ opens up and has vertex at $(0,-3)$. They are clearly tangent at $(0,-3)$. The only question is whether the parabola lies outside the circle except at $(0,-3)$ or if part of it is inside the circle. Since



$$(\pm 1)^2-3=-2$$

the point $(\pm 1,-2)$ is on the parabola. But $(\pm 1)^2+(-2)^2=5<9$, so that $(\pm 1,-2)$ is inside the circle. So there must be 3 intersection points.

Alternative solution:

Solving the equations $x^2=9-y^2$ and $x^2=y+3$ simultaneously gives

$$9-y^2=y+3\Rightarrow y^2+y-6=0\Rightarrow (y+3)(y-2)=0\Rightarrow y=-3 \text{ or } y=2$$

So the three intersection points are $(0,-3)$ and $\left(\pm\sqrt{5},2\right)$.

Answer is (D).

7. There were 8 pitas shared equally between 3 people so each got $\frac{8}{3}$ of a pita. Host A ate $\frac{8}{3}$ pitas and sold $3 - \frac{8}{3} = \frac{1}{3}$ pita to the traveler. Host B ate $\frac{8}{3}$ pitas and sold $5 - \frac{8}{3} = \frac{7}{3}$ pitas to the traveler. Host B sold 7 times as much to the traveler as host A. He should get 7 gold coins and host A should get 1 gold coin. B should get another 2 gold coins.

Answer is (D).

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8. There are $9^3 = 729$ possible choices for A, B, and C, but solving for C to give $C = A^B + B^A - 10A$ brings the number down to $9^2 = 81$. Consider the nine possible values for A:

- If $A = 1$ then $C = B - 9 \geq 1 \Rightarrow B \geq 10$, so there are no solutions.
- If $A = 2$ then $C = 2^B + B^2 - 20$. If $B = 3$ then $C = -3$ (too small), while if $B = 4$ then $C = 12$ (too big) so there are no solutions.
- If $A = 3$ then $C = 3^B + B^3 - 30$. If $B = 2$ then $C = -13$ (too small), while if $B = 3$ then $C = 24$ (too big) so there are no solutions.
- If $A = 4$ then $C = 4^B + B^4 - 40$. If $B = 2$ then $C = -8$ (too small), while if $B = 3$ then $C = 105$ (too big) so there are no solutions.
- If $A = 5$ then $C = 5^B + B^5 - 50$. If $B = 1$ then $C = -44$ (too small), if $B = 2$ then $C = 7$ (which works). $A = 5$, $B = 2$, $C = 7$ is a solution.
- If $A = 6$ then $C = 6^B + B^6 - 60$. If $B = 1$ then $C = -53$ (too small), while if $B = 2$ then $C = 40$ (too big) so there are no solutions.
- If $A = 7$ then $C = 7^B + B^7 - 70$. If $B = 1$ then $C = -62$ (too small), while if $B = 2$ then $C = 107$ (too big) so there are no solutions.
- If $A = 8$ then $C = 8^B + B^8 - 80$. If $B = 1$ then $C = -71$ (too small), while if $B = 2$ then $C = 240$ (too big) so there are no solutions.
- If $A = 9$ then $C = 9^B + B^9 - 90$. If $B = 1$ then $C = -80$ (too small), while if $B = 2$ then $C = 503$ (too big) so there are no solutions.

There is only one possible solution, $A = 5$, $B = 2$, and $C = 7$. Adding the digits gives $5 + 2 + 7 = 14$.

Answer is (D).

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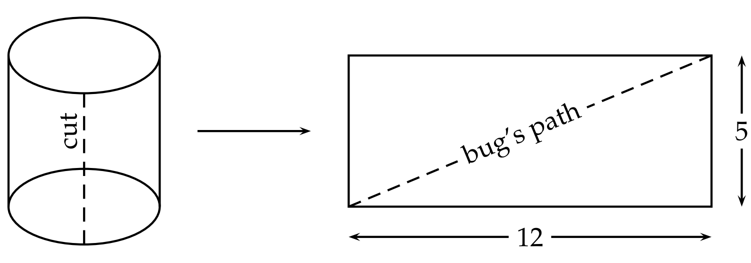
9. Cut the can along a straight line from a point on the bottom of the can (where the bug starts) to the point directly above it (where the bug ends) and unroll the can to obtain a rectangle. The bug's path is the straight line from one corner to the diagonally opposite corner, as shown.



The length of this line is

$$\sqrt{5^2 + 12^2} = 13$$

Answer is (B).

10. Count the squares:

$$\begin{aligned}
\text{number of squares of side } 1 &= 9 \\
\text{number squares of side } 2 &= 4 \\
\text{number squares of side } 3 &= 1 \\
\text{number of squares of side } \sqrt{2} &= 4 \\
\text{number of squares of side } \sqrt{5} &= 2
\end{aligned}$$

Adding gives $9+4+1+4+2=20$.

Answer is (E).

Junior Final, Part B

1. Alexa performs the following sequence of operations on the output at each step beginning with the original amount: divide by 2; subtract 13; divide by 2; subtract 1; and she arrives at the number 3. Reversing this process beginning with 3, she obtains the original number $((3+1)\cdot 2+13)\cdot 2=42$.

   Alternative solution:

   Let x be the number of jellybeans. The x satisfies the equation

$$\tfrac{1}{2}x-13-\left[\tfrac{1}{2}\left(\tfrac{1}{2}x-13\right)+1\right]=3 \Rightarrow \frac{x}{4}-\frac{15}{2}=3 \Rightarrow x=42$$

Answer: 42 jellybeans

2. The elastic band consists of three straight sections and three curved sections. Each of the straight sections has length twice the radius of the coins for a total of $6r$. Each of the curved sections is a part of a circle subtending an angle of

$$\theta=120^\circ=\frac{2\pi}{3}\text{ radians}$$

   Hence, the three curved sections together form a complete circle. Therefore, the total length of the curved sections is the circumference of the circle $C=2\pi r$. Since $r=1$, the length of the elastics is $6+2\pi$.

Answer: The length is $6+2\pi$.

3. Clearly, both x and y must be greater than 6. Further, if $x = 12$, then $y = 12$. First assume that $x < y$. In this case, x can only take integer values from $x = 7$ to $x = 12$, since if $x > 12$, then $y < x$ (as seen below). Substituting these values into the equation above gives

$$\begin{aligned} x = 7 &\Rightarrow y = 42 \\ x = 8 &\Rightarrow y = 24 \\ x = 9 &\Rightarrow y = 18 \\ x = 10 &\Rightarrow y = 15 \\ x = 12 &\Rightarrow y = 12 \end{aligned}$$

Expressing the solutions as ordered pairs (x, y) gives

$$(7, 42), (8, 24), (9, 18), (10, 15), (12, 12)$$

Interchanging the roles of x and y gives four more possibilities as

$$(15, 10), (18, 9), (24, 8), (42, 7)$$

Alternative solution:
Multiply both sides of the equation by $6xy$ to obtain the equation

$$6y + 6x = xy$$

Subtract $6y + 6x$ from both sides of the equation and add 36 to both sides, to give

$$xy - 6y - 6x + 36 = 36 \Rightarrow y\left(x-6\right) - 6\left(x-6\right) = \left(x-6\right)\left(y-6\right) = 36$$

Observe that $36 = 2^2 \cdot 3^2$, so that 36 can be factored as

$$36 = 1 \cdot 36 = 2 \cdot 18 = 3 \cdot 12 = 4 \cdot 9 = 6 \cdot 6$$

This gives the following values for x and y:

$$\begin{aligned} x - 6 = 1 \ \&\ y - 6 = 36 &\Rightarrow x = 7 \ \&\ y = 42 \\ x - 6 = 2 \ \&\ y - 6 = 18 &\Rightarrow x = 8 \ \&\ y = 24 \\ x - 6 = 3 \ \&\ y - 6 = 12 &\Rightarrow x = 9 \ \&\ y = 18 \\ x - 6 = 4 \ \&\ y - 6 = 9 &\Rightarrow x = 10 \ \&\ y = 15 \\ x - 6 = 6 \ \&\ y - 6 = 6 &\Rightarrow x = 12 \ \&\ y = 12 \end{aligned}$$

By symmetry there are four other possible solutions with the x and y values reversed. Hence, the possible solutions, as ordered pairs (x, y), are:

$$(7, 42), (8, 24), (9, 18), (10, 15), (12, 12), (15, 10), (18, 9), (24, 8), (42, 7)$$

Answer: The solutions are $(7, 42), (8, 24), (9, 18), (10, 15), (12, 12), (15, 10), (18, 9), (24, 8), (42, 7)$**.**

4. Let x represent the number of sheep bought by a man and y be the number of sheep bought by his wife. Then each spent x^2 and y^2 yahoos, respectively. The husband spent 5 googles = $5 \times 21 = 105$ yahoos more than his wife, so for each couple $x^2 - y^2 = (x-y)(x+y) = 105 = 3 \times 5 \times 7$. These must all be integers with $0 < x - y < x + y$, so the only possibilities are

$x-y$	$x+y$	x	y
1	105	53	52
3	35	19	16
5	21	13	8
7	15	11	4

Since "Archie bought 11 more sheep than Amanda," Archie bought 19 sheep and Amanda bought 8. Since "Caroline bought 5 more sheep than Chuck," Caroline bought 16 sheep and Chuck bought 11. Since "Dave bought 9 more sheep than Bernadette," Dave bought 13 and Bernadette bought 4. This gives

x	y
Bruce: 53	Diane: 52
Archie: 19	Caroline: 16
Dave: 13	Amanda: 8
Chuck: 11	Bernadette: 4

(by elimination Bruce and Diane are married.) Amanda is married to Dave. Bruce and Diane must be rich and *really* like sheep...

Answer: Amanda is married to Dave.

---

5. Let D' be the base of the perpendicular from A' to the side DA. The area of the unshaded and blue regions are equal, so that the area of each of the unshaded, blue, and red regions is one third the area of the square $ABCD$ which is $\frac{12}{3} = 4$. It follows that the area of the square $AD'A'F$ is 8 and, so, the side length of the square is

$$\left|\overline{AD'}\right| = \left|\overline{AF}\right| = \sqrt{8}$$

Further, the side length of the square $ABCD$ is

$$\left|\overline{AD}\right| = \left|\overline{AB}\right| = \sqrt{12}$$

The diagonals of the two squares are

$$\left|\overline{AC}\right| = \sqrt{\left|\overline{AB}\right|^2 + \left|\overline{AD}^2\right|} = \sqrt{2\left(\sqrt{12}\right)^2} = 2\sqrt{6}$$

$$\left|\overline{AA'}\right| = \sqrt{\left|\overline{AD'}\right|^2 + \left|\overline{D'A'}^2\right|} = \sqrt{2\left(\sqrt{8}\right)^2} = 4$$

Then the required distance is the difference between the lengths of the diagonals of the squares $ABCD$ and $AD'A'F$ which is

$$\left|\overline{A'C}\right| = 2\sqrt{6} - 4 = 2\left(\sqrt{6} - 2\right)$$

Problem 5 continued ...

...Problem 5 continued

Alternative solution:

Let E be the base of the perpendicular from A' to the side BC and F be the base of the perpendicular from A' to side AB. It is clear that triangle $A'CE$ is similar to triangle AFA'. Further,

$$\left|\overline{A'E}\right| = \left|\overline{DD'}\right| = \sqrt{12} - \sqrt{8} = \sqrt{2}\left(\sqrt{6} - 2\right)$$

Hence,

$$\frac{\left|\overline{A'C}\right|}{\left|\overline{A'E}\right|} = \frac{\left|\overline{AC}\right|}{\left|\overline{AB}\right|} = \frac{2\sqrt{6}}{\sqrt{12}} = \sqrt{2} \Rightarrow \left|\overline{A'C}\right| = \sqrt{2}\left[\sqrt{2}\left(\sqrt{6} - 2\right)\right] = 2\left(\sqrt{6} - 2\right)$$

Alternative solution:

Determining the diagonal of the triangle $A'CE$ using Pythagoras' theorem gives

$$\left|\overline{A'C}\right| = \sqrt{2\left|\overline{DD'}\right|^2} = \sqrt{2\left[\sqrt{2}\left(\sqrt{6} - 2\right)\right]^2} = 2\left(\sqrt{6} - 2\right)$$

Answer: The distance is $2(\sqrt{6} - 2)$**.**

Senior Final, Part A

1. The set S consists of the numbers: 319, 328, 337, 346, 355, 364, 373, 382, 391. Since the first digit is 3 in each case, it is only necessary to identify only those numbers for which the final two digits have a product of more than 20. These numbers are: 337, 346, 355, 364, 373. So $\dfrac{5}{9}$ of the elements of S have digits with a product greater than 60.

Answer is (D).

2. Writing the equation as
$$(4^x)^2 - 3 \cdot 4^x + 2 = (4^x - 1)(4^x - 2) = 0$$
shows that the solutions are given by $4^x = 1$ and $4^x = 2$. The first requires that $x = 0$ and the second requires that $x = \frac{1}{2}$. Hence, the sum of the solutions is $0 + \frac{1}{2} = \frac{1}{2}$.

Answer is (B).

3. The elastic band consists of three straight sections and three curved sections. Each of the straight sections has length twice the radius of the coins for a total of $6r$. Each of the curved sections is a part of a circle subtending an angle of
$$\theta = 120^\circ = \frac{2\pi}{3} \text{ radians}$$
The formula for arc length is $s = r\theta$ where θ is in radians, so the total for the curved sections is $2\pi r$. Since $r = 1$, the length of the elastics is $6r + 2\pi$.

Answer is (C).

4. Let $\{a_n\}$ be the sequence. The description states that

$$a_n a_{n-1} = a_n - 1 \Rightarrow a_n = -\frac{1}{a_{n-1} - 1}$$

The first few terms are

$$a_1 = 5,\ a_2 = -1/4,\ a_3 = 4/5,\ a_4 = 5$$

and the series repeats:

$$a_{3n+1} = 5,\ a_{3n+2} = -1/4,\ a_{3n+3} = 4/5,\ n = 0,\ 1,\ 2,\ \cdots$$

$2011 = 3 \times 670 + 1$ so $a_{2011} = a_1 = 5$.

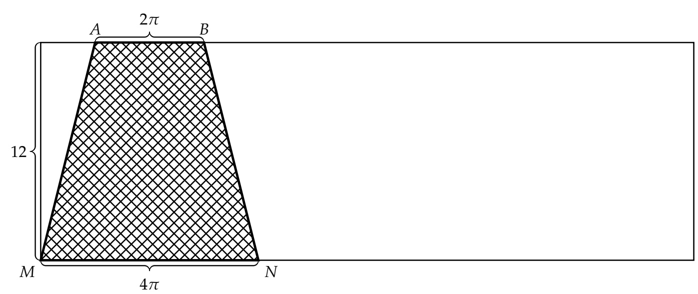
Answer is (A).

---

5. The covered portion of the cylinder consists of three sections: two circular sectors and the covered portion of the sides. The area of a circular sector of radius r and central angle θ, in radians, is $A = \frac{1}{2}r^2\theta$. The radius of both circular sectors is $r = 6$. So the combined area of the circular sectors is

$$A_{\text{sectors}} = \frac{1}{2}\left(\frac{\pi}{3}\right)(6)^2 + \frac{1}{2}\left(\frac{2\pi}{3}\right)(6)^2 = 18\pi$$

To determine the area of the covered portion of the sides, unroll the cylinder to obtain a rectangle:



The area of the covered portion of the sides is the area of the trapezoid $ABNM$. This is

$$A_{\text{sides}} = (12)\left(\frac{2\pi + 4\pi}{2}\right) = 36\pi$$

Hence, the total area of the covered portion of the cylinder is

$$A = A_{\text{sectors}} + A_{\text{sides}} = 18\pi + 36\pi = 54\pi$$

Answer is (D).

6. Label each path with the proportion of hikers that take it, as shown in the diagram. Then read off the proportion of hikers arriving at each destination:

$$\begin{array}{cc} A & \dfrac{1}{16} \\ B & \dfrac{1}{16} \\ C & \dfrac{1}{8}+\dfrac{1}{4}+\dfrac{1}{8}=\dfrac{1}{2} \\ D & \dfrac{1}{8}+\dfrac{1}{4}=\dfrac{3}{8} \end{array}$$

Answer is (B).

---

7. Each cup must contain at least 2 marbles for a total of 6 marbles. If these are assigned at the start, there are 3 marbles left to divide among the 3 cups. Doing this in ascending order of largest number assigned to each cup gives the assignments: $(0,0,3)$, $(0,1,2)$, and $(1,1,1)$. The first can be arranged in three ways: $(0,0,3)$, $(0,3,0)$, $(3,0,0)$. The second can be arranged in six ways: $(0,1,2)$, $(0,2,1)$, $(1,0,2)$, $(1,2,0)$, $(2,0,1)$, $(2,1,0)$. There is only one way to arrange the third: $(1,1,1)$. Therefore, there is a total of 10 ways to divide the nine marbles into the three cup.

Alternative solution:

The assignment of the 3 marbles to the 3 cups can be represented using an arrangement of $5 = 3 + (3-1)$ spaces to be filled by a $\star$, representing a marble, or a vertical line, $|$, representing a boundary between cups. Note that only two vertical lines are necessary, since this give three divisions. For example

$$\begin{array}{cccccl} \star & \star & | & | & \star & \text{represents } (2,0,1) \\ \star & | & \star & | & \star & \text{represents } (1,1,1) \end{array}$$

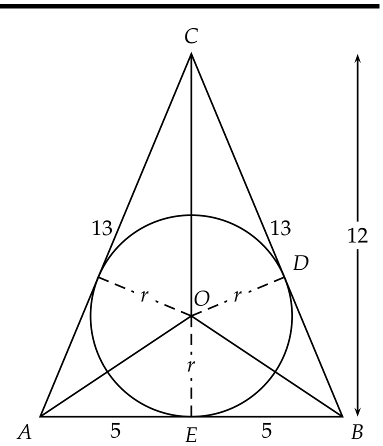
The number of possible arrangements is the number of ways to select the two spaces for the vertical lines from the five spaces, or the three spaces for the $\star$'s. This is

$${}_5C_2 = {}_5C_3 = \frac{5!}{3!2!} = 10$$

Answer is (A).

---

8. The radius r of the sphere is the radius of the circle inscribed in a triangle with sides 10, 13, and 13, as shown in the diagram. The area of this triangle is $\frac{1}{2}(10 \times 12) = 60$. Alternatively, partition the triangle into three triangles by drawing lines from the incentre to each vertex. The areas of these triangles are $\frac{1}{2}(10 \times r) = 5r$, $\frac{1}{2}(13 \times r) = \frac{13}{2}r$, and $\frac{1}{2}(13 \times r) = \frac{13}{2}r$. Adding gives $18r = 60 \Rightarrow r = \frac{10}{3}$.



Problem 8 continued ...

...Problem 8 continued

Alternative solution:

Triangles BOD and BOE are congruent. Hence, $|\overline{BD}| = |\overline{BE}| = 5$. This means that $|\overline{DC}| = 8$. Then applying Pythagoras' theorem to triangle COD gives

$$|\overline{CO}|^2 = r^2 + 8^2 = r^2 + 64$$

But,

$$|\overline{CO}| + r = 12 \Rightarrow |\overline{CO}|^2 = (12 - r)^2 = 144 - 24r + r^2$$

Equating the two expressions for $|\overline{CO}|^2$ and solving for r gives

$$r^2 + 64 = 144 - 24r + r^2 \Rightarrow 24r = 80 \Rightarrow r = \tfrac{10}{3}$$

as before.

Answer is (C).

---

9. Count the squares:

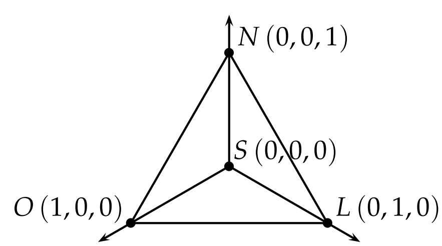
$$\begin{aligned} \text{number of squares of side } 1 &= 4 \\ \text{number squares of side } 2 &= 2 \\ \text{number of squares of side } \sqrt{2} &= 2 \\ \text{number of squares of side } \sqrt{5} &= 2 \end{aligned}$$

Adding gives $4 + 2 + 2 + 2 = 10$

Answer is (D).

---

10. The distance of any of the objects from the observer is not important—only the angles which they make. Consequently, they can be placed anywhere that is convenient. Using a standard xyz-coordinate system can put the ship at the origin $S(0,0,0)$ and the observer at $O(1,0,0)$. If the North Star is at $N(0,0,1)$, then $\angle NOS = 45°$. If the lighthouse is at $L(0,1,0)$ then $\angle SOL = 45°$. The triangle $\triangle NOL$ is equilateral so $\angle NOL = 60°$.



Alternative solution:

Using the spherical Law of Cosines,

$$\cos c = \cos a \cos b + \sin a \sin b \cos \gamma$$

with $a = b = \frac{\pi}{4}$, $\gamma = \frac{\pi}{2}$, and $c = \angle NOL$ gives

$$\cos c = \cos \frac{\pi}{4} \cos \frac{\pi}{4} + \sin \frac{\pi}{4} \sin \frac{\pi}{4} \cos \frac{\pi}{2} = \frac{1}{2} \Rightarrow c = \frac{\pi}{3}$$

as in the previous solution.

Answer is (D).

Senior Final, Part B

1. (a) Write the equation as

$$3 \cdot 5^m = n^2 - 1 = (n+1)(n-1)$$

Thus, $3 \cdot 5^m$ must factor into two factors that differ by 2. Obviously,

$$\begin{aligned} m = 0 &\Rightarrow 3 = (n+1)(n-1) \Rightarrow n = 2 \\ m = 1 &\Rightarrow 3 \cdot 5 = (n+1)(n-1) \Rightarrow n = 4 \end{aligned}$$

$\boxed{\textbf{Answer: Two solutions are } (m,n) = (0,2) \textbf{ and } (m,n) = (1,4).}$

(b) To show that these are the only solutions factor $3 \cdot 5^m$ as $3 \cdot 5^m = 3 \cdot 5^a \cdot 5^b$. For $a > b$ the positive difference between the factors is

$$3 \cdot 5^a - 5^b = 5^b \left(3 \cdot 5^{a-b} - 1\right) = 2 \Rightarrow b = 0 \;\&\; a = 0 \Rightarrow m = 0$$

and for $a \leq b$ the positive difference between the factors is

$$5^b - 3 \cdot 5^a = 5^a \left(5^{b-a} - 3\right) = 2 \Rightarrow a = 0 \;\&\; b = 1 \Rightarrow m = 1$$

Hence, the only solutions are $m = 0$ and $n = 2$, and $m = 1$ and $n = 4$.

$\boxed{\textbf{Answer: See proof above.}}$

---

2. Let the hats be numbered 1, 2, 3, and 4. Indicate which hat is picked up by which woman as an ordered 4-tuple; for example, $(1,2,3,4)$ indicates that each woman got her own hat. The $4! = 24$ permutations of this 4-tuple are:

$(1,2,3,4)$	$(2,1,3,4)$	$(3,1,2,4)$	$\boxed{(4,1,2,3)}$
$(1,2,4,3)$	$\boxed{(2,1,4,3)}$	$\boxed{(3,1,4,2)}$	$(4,1,3,2)$
$(1,3,2,4)$	$(2,3,1,4)$	$(3,2,1,4)$	$(4,2,1,3)$
$(1,3,4,2)$	$\boxed{(2,3,4,1)}$	$(3,2,4,1)$	$(4,2,3,1)$
$(1,4,2,3)$	$\boxed{(2,4,1,3)}$	$\boxed{(3,4,1,2)}$	$\boxed{(4,3,1,2)}$
$(1,4,3,2)$	$(2,4,3,1)$	$\boxed{(3,4,2,1)}$	$\boxed{(4,3,2,1)}$

The 4-tuples in which none of the numbers is in its correct position represents a situation in no woman picked up her own hat. Each of the nine such 4-tuple is boxed in the table above. So the probability that no woman picks up her own hat is

$$\frac{9}{24} = \frac{3}{8}$$

Alternative solution:

There are three ways the second hat can be picked up by the first woman and have none of the other three hats picked up by its owner, represented as: $(2,1,4,3)$, $(2,4,1,3)$, and $(2,3,4,1)$. (See the table above.) Similarly, there are three ways in which the third hat is picked up by the first woman and three ways in which the fourth hat is picked up by the first first woman. This gives a total of 9 ways.

Problem 2 continued ...

...Problem 2 continued

Alternative solution:

As noted above there are $4! = 24$ permutations four hats. It is desired to count those permutations in which no hat is picked up by its owner. Do this by first counting those permutations in which at least one hat is picked up by its owner. There is 1 permutation in which all four hats are mapped to picked up by their owners. If three hats are picked by their owners, then the fourth is also picked up by its owner. So there are no permutations in which exactly three are picked by their owners. There are 6 ways to choose two of the hats to be picked up by their owners, and, once these are chosen, the other two can only be picked up in one way. So there are 6 permutations in which exactly two hats are picked up by their owners. There are 4 ways to choose one hat to be picked up by its owner. Once this hat is chosen the other three can be picked up in two ways so that none of them are picked up by their owners. So, there are 8 permutations in which exactly one hat is picked up by its owner. This gives a total of $1 + 6 + 8 = 15$ permutations in which at least one hat is picked up by its owner. Hence, there are $24 - 15 = 9$ permutations in which no hat is picked up by its owner.

Answer: The probability is $\frac{3}{8}$.

---

3. The problem does not specify anything about the speed of the current in the river. Because of this two solutions are given based on the assumptions:

   I) the barge is self-powered with a speed of one kilometre per hour and there is no current

   II) the barge is un-powered and drifts with the current, which must be one kilometre per hour

   In any other case, the answer will depend on the speed of the current. (See the General Solution below.)

   Assumption I

   Let v_b be the speed of the barge and v_d be the speed of the duck. Let t_1 be the time it takes for the duck to swim to the back of the barge and t_2 be the time it takes for it to swim back to the front. As the duck swims to the back of the barge, the barge travels $v_1 t_1$ so the duck travels

$$v_d t_1 = L - v_b t_1$$

   As the duck swims back to the front of the barge, the barge travels $v_1 t_2$ so the duck travels

$$v_d t_2 = L + v_b t_2$$

   The total time it takes the barge moves its entire length it $t_1 + t_2$, so

$$v_b(t_1 + t_2) = L$$

   Substituting this expression for L into the first two equations gives

$$v_d t_1 = v_b(t_1 + t_2) - v_b t_1 = v_b t_2 \Rightarrow \frac{v_b}{v_d} = \frac{t_1}{t_2}$$
$$v_d t_2 = v_b(t_1 + t_2) + v_b t_2 = v_b t_1 + 2v_b t_2$$
$$v_d = v_b \frac{t_1}{t_2} + 2v_b = \frac{v_b^2}{v_d} + 2v_b \Rightarrow v_d^2 - 2v_b v_d - v_b^2 = 0$$

   This is a quadratic in v_d. Using the quadratic formula, and recalling that $v_b = 1$, gives

$$v_d = \frac{2 \pm \sqrt{2^2 + 4}}{2} = \left(1 \pm \sqrt{2}\right)$$

   Since $v_d > 0$, take the positive root, so that $v_d = \left(1 + \sqrt{2}\right)$.

Problem 3 continued ...

...Problem 3 continued

Assumption II

Let v_c be the speed of the current and, hence, the barge, and v_d be the speed of the duck in still water. Let t_1 be the time it takes for the duck to swim to the back of the barge and t_2 be the time it takes for it to swim back to the front. When the duck swims to the back of the barge it swims against the current at a speed of $v_d - v_c$ relative to the land. During this time the barge travels $v_c t_1$, and the duck travels

$$(v_d - v_c)\, t_1 = L - v_c t_1 \Rightarrow v_d t_1 = L$$

When the duck swims back to the front of the barge it swims with the current at a speed of $v_d + v_c$ relative to the land. During this time the barge travels $v_c t_2$, and the duck travels

$$(v_d + v_c)\, t_2 = L + v_c t_2 \Rightarrow v_d t_2 = L$$

Adding the two equations above gives

$$v_d\,(t_1 + t_2) = 2L$$

The total time it takes the barge moves its entire length it $t_1 + t_2$, so

$$v_c(t_1 + t_2) = L$$

Substituting this expression for L into the previous equation gives

$$v_d\,(t_1 + t_2) = 2v_c\,(t_1 + t_2) \Rightarrow v_d = 2v_c$$

Since the current is one kilometre per hour, the duck swims at a speed of two kilometres per hour.

General solution:

Let v_c be the speed of the current, v_b the speed of the barge in still water, and v_d be the speed of the duck in still water. Let t_1 be the time it takes for the duck to swim to the back of the barge and t_2 be the time it takes for it to swim back to the front. Then, the speed of the barge relative to the land is $v_b + v_c$. When the duck swims to the back of the barge it swims against the current at a speed of $v_d - v_c$ relative to the land. During this time the barge travels $(v_b + v_c)\, t_1$, and the duck travels

$$(v_d - v_c)\, t_1 = L - (v_b + v_c)\, t_1 \Rightarrow (v_d + v_b)\, t_1 = L \Rightarrow t_1 = \frac{L}{v_d + v_b}$$

When the duck swims back to the front of the barge it swims with the current at a speed of $v_d + v_c$ relative to the land. During this time the barge travels $(v_b + v_c)\, t_2$, and the duck travels

$$(v_d + v_c)\, t_2 = L + (v_b + v_c)\, t_2 \Rightarrow (v_d - v_b)\, t_2 = L \Rightarrow t_2 = \frac{L}{v_d - v_b}$$

The total time it takes the barge to move its entire length is $t_1 + t_2$, so

$$(v_b + v_c)\,(t_1 + t_2) = L \Rightarrow t_1 + t_2 = \frac{L}{v_b + v_c}$$

Combining the first two equations with the third gives

$$t_1 + t_2 = \frac{L}{v_b + v_c} = \frac{L}{v_d + v_b} + \frac{L}{v_d - v_b}$$

Canceling the L and adding the fractions on the righthand side gives

$$\frac{1}{v_b + v_c} = \frac{1}{v_d + v_b} + \frac{1}{v_d - v_b} = \frac{2v_b}{v_d^2 - v_b^2}$$

...Problem 3 continued

Clearing the fractions to obtain an equation for v_d gives

$$v_d^2 - 2c_d\left(v_b + v_c\right) - v_b^2 = 0$$

Solving this quadratic equation for v_d gives

$$v_d = \frac{2\left(v_b + v_c\right) \pm \sqrt{4\left(v_b + v_c\right)^2 + 4v_b^2}}{2} = v_b + v_c \pm \sqrt{\left(v_b + v_c\right)^2 + v_b^2}$$

Take the positive root above so that $v_d > 0$. So the solution for any current is

$$v_d = v_b + v_c + \sqrt{\left(v_b + v_c\right)^2 + v_b^2}$$

Under Assumption I the speed of the current is $v_c = 0$, so that

$$v_d = v_b + \sqrt{2v_b^2} = 1 + \sqrt{2} \quad \left(v_b = 1\right)$$

Under Assumption II the speed of the barge in still water is $v_b = 0$, so that

$$v_d = v_c + \sqrt{v_c^2} = 2 \quad \left(v_c = 1\right)$$

Answer: The duck's speed is either $\left(1 + \sqrt{2}\right)$ kilometres per hour or 2 kilometres per hour.

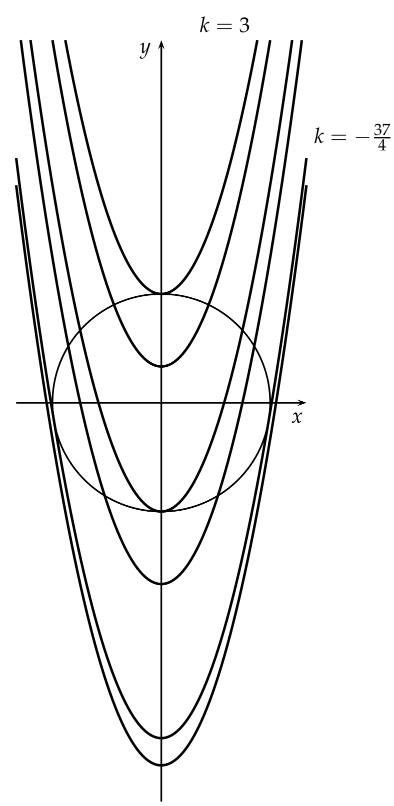
4. The diagram shows the circle and the parabola for several values of k. The point(s) of intersection of the two curves are determined by solving the two equations simultaneously. Solving the equation of the parabola for $x^2 = y - k$ and substituting into the equation of the circle gives

$$y - k + y^2 = 9 \Rightarrow y^2 + y - k - 9 = 0$$

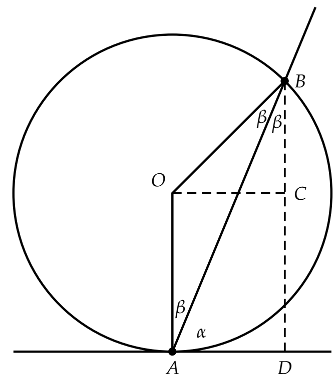
Solving for y

$$y = \frac{-1 \pm \sqrt{1 - 4\left(-k - 9\right)}}{2} = \frac{-1 \pm \sqrt{37 + 4k}}{2}$$

There is only one solution for y when the parabola is tangent to the circle at the point of intersection. This is the case when $k = -\dfrac{37}{4}$. From the diagram it is clear that the parabola does not intersect the circle when $k < -\dfrac{37}{4}$. Further, the diagram shows that the parabola does not intersect the circle when $k > 3$. This can also be determined by requiring that $x^2 = y - k > 0$. Hence, there are no points of intersection of the circle $x^2 + y^2 = 9$ and the parabola $y = x^2 + k$ when $k < -\dfrac{37}{4}$ or $k > 3$.



Answer: There are no points of intersection when $k < -\dfrac{37}{4}$ or $k > 3$.



5. In the diagram $\alpha = 67\frac{1}{2}^\circ$, and the tangent at A is perpendicular to the radius, so that $\beta = 90^\circ - \alpha = 22\frac{1}{2}^\circ$. Observing that triangle AOB is isosceles with $|\overline{AO}| = |\overline{BO}| = 1$, gives $\angle BAO = \angle ABO = \beta$. Draw the line BD perpendicular to the tangent at D and the line OC perpendicular to the line BD at C. Then triangle ABD is a right triangle with $\angle ABD = 90^\circ - \alpha = 22\frac{1}{2}^\circ = \beta$. Hence, $\angle OBD = 45^\circ$ which means that triangle OBC is an isosceles right triangle with

$$|\overline{BC}| = |\overline{OC}| = \frac{1}{\sqrt{2}}$$

Therefore, triangle ABD has perpendicular sides

$$\begin{aligned} |\overline{AD}| &= \frac{1}{\sqrt{2}} \\ |\overline{BD}| &= |\overline{CD}| + |\overline{BC}| = |\overline{OA}| + \frac{1}{\sqrt{2}} = 1 + \frac{1}{\sqrt{2}} \end{aligned}$$

and the length of the segment $\overline{AB}$ is given by Pythagoras' theorem as

$$\begin{aligned} |\overline{AB}| &= \sqrt{|\overline{AD}|^2 + |\overline{BD}|^2} = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(1 + \frac{1}{\sqrt{2}}\right)^2} \\ &= \sqrt{\frac{1}{2} + 1 + \frac{2}{\sqrt{2}} + \frac{1}{2}} = \sqrt{2 + \sqrt{2}} \end{aligned}$$

Alternative solution:

In the diagram, $\angle AOB = 180^\circ - 2\beta = 2\alpha = 135^\circ$. Then, using the Law of Cosines the length of the chord $\overline{AB}$ is given by

$$\begin{aligned} |\overline{AB}|^2 &= |\overline{AO}|^2 + |\overline{BO}|^2 - 2\,|\overline{AO}|\,|\overline{B}|^2 \cos 135^\circ \\ &= 1 + 1 - 2\left(-\frac{\sqrt{2}}{2}\right) = 2 + \sqrt{2} \end{aligned}$$

Therefore, the length of the segment is $|\overline{AB}| = \sqrt{2 + \sqrt{2}}$.

Alternative solution:

Since $\alpha = 67\frac{1}{2}^\circ$, $\beta = 90^\circ - \alpha = 22\frac{1}{2}^\circ$. The length of the chord is $2\cos\beta$. Using the half-angle formula

$$\cos\left(\frac{1}{2}\theta\right) = \sqrt{\frac{\cos\theta + 1}{2}}$$

if $\theta = 45^\circ$, then $\beta = \frac{1}{2}\theta$ and

$$\cos\beta = \sqrt{\frac{\cos\theta + 1}{2}} = \sqrt{\frac{\frac{\sqrt{2}}{2} + 1}{2}} = \sqrt{\frac{\sqrt{2} + 2}{4}} \Rightarrow 2\cos\beta = \sqrt{\sqrt{2} + 2}$$

The length of the segment is $|\overline{AB}| = \sqrt{2 + \sqrt{2}}$.

Answer: The length is $|\overline{AB}| = \sqrt{2 + \sqrt{2}}$.