BRITISH COLUMBIA SECONDARY SCHOOL MATHEMATICS CONTEST, 2010

Solutions

Junior Preliminary

1. Add pairs of numbers in the given expression, beginning with the outermost pair and working inwards. Since the sum of each pair is 1 and there are evidently 50 such pairs, the value of the expression is 50.

   Alternative solution

   Write the sum as

$$\begin{aligned} 100-1+98-3+96-5+\cdots+4-97+2-99 &= (100+98+\cdots+2)-(1+3+\cdots+99) \\ &= 2(50+49+\cdots+1)-\Big(\underbrace{1+3+\cdots+99}_{\text{50 terms}}\Big) \\ &= 2\left[\tfrac{1}{2}50(51)\right]-50^2 = 50\times 51-50^2 = 50 \end{aligned}$$

Answer is (B).

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2. Multiplying both the numerator and the denominator by 12 and simplifying, we find

$$\frac{\frac{1}{2}+\frac{1}{3}}{\frac{1}{3}+\frac{1}{4}} = \frac{6+4}{4+3} = \frac{10}{7}$$

Answer is (D).

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3. The units digit of 1855 is 5, so the units digit of the two-digit number must be 5 and the units digit of its reversal must be odd (or *vice-versa*). Trying successively, 15, 35, ... we find success at 35. It follows that the two numbers are 35 and 53, and the sum of the digits is 8. Note that $50\times 40 > 1855$, so that the digit that is not 5 must be less than 4. So it is only necessary to test 15 and 35.

Answer is (A).

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4. Let R be the radius of the large circle and r the radius of the small circles, then

$$2\pi R = 3$$

and $r=\frac{1}{3}R$. Hence, the total perimeter is

$$P = 2\pi R + 7(2\pi r) = 2\pi R + 7\left[2\pi\left(\tfrac{1}{3}R\right)\right] = 2\pi R + \tfrac{7}{3}(2\pi R) = 3+7 = 10$$

Answer is (E).

5. With the obvious notation and units, we have $3p + 2a = 92$ and $2p + 3a = 83$. Adding these equations and dividing by 5, we find $p + a = \frac{175}{5} = 35$ (cents).

Answer is (A).

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6. Since $2.\overline{3} = \frac{7}{3}$ and $3.\overline{6} = \frac{11}{3}$, the product is

$$2.\overline{3} \times 3.\overline{6} = \frac{7}{3} \times \frac{11}{3} = \frac{77}{9} = 8\frac{5}{9} = 8.\overline{5}$$

Answer is (A).

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7. Since $2010 = 2 \cdot 3 \cdot 5 \cdot 67$, by the fundamental theorem of arithmetic all of these primes must be factors of the given product of even numbers. This first occurs when one of the factors in the product is 134, and this number lies between 131 and 141.

Answer is (D).

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8. Any integers a and b satisfying the equation must also satisfy the inequalities $-5 \leq a \leq 5$ and $-2 \leq b \leq 2$, otherwise the left-hand side is too large. Looking at all pairs (a, b) satisfying these inequalities, it is clear that there are six which also satisfy the equation; namely, $(-3, -2)$, $(-3, 2)$, $(3, -2)$, $(3, 2)$, $(5, 0)$ and $(-5, 0)$.

Alternative solution

First note that $3^2 + 4^2 = 9 + 16 = 5^2 = 25$ is the only Pythagorean triple for which the sum of the squares is 25, and, also that, $5^2 + 0^2 = 25$. Hence, writing the given equation as

$$a^2 + 4b^2 = a^2 + (2b)^2 = 25$$

gives $a^2 = 9 \Rightarrow a = \pm 3$ and $(2b)^2 = 16 \Rightarrow 2b = \pm 4 \Rightarrow b = \pm 2$, or $a^2 = 25 \Rightarrow a = \pm 5$ and $(2b)^2 = 0 \Rightarrow b = 0$. This gives the six possible pairs listed above.

Answer is (C).

9. The sum of all the positive integers less than 2010 is

$$\frac{2009 \times 2010}{2} = 3\left(\frac{2009 \times 670}{2}\right)$$

The sum of all the positive integers less than 2010 which are divisible by three is

$$3 + 6 + 9 + \cdots + 2007 = 3\,(1 + 2 + 3 + \cdots 669) = 3\left(\frac{669 \times 670}{2}\right)$$

The required answer is

$$3 \times 670\left(\frac{2009 - 669}{2}\right) = 3 \times 670\left(\frac{1340}{2}\right) = 3 \times 670^2 = 1346700$$

Alternative solution
The sum of the integers less than 2010 that are not divisible by 3 is

$$(1 + 2) + (4 + 5) + (7 + 8) + \cdots + (2008 + 2009) = 3 + 9 + 15 + \cdots + 4017 = 3\,(1 + 3 + 5 + \cdots + 1339)$$

This is an arithmetic progression of 670 terms with starting value 1 and difference 2. The sum is

$$670 \times \frac{1 + 1339}{2} = 670^2$$

So the answer is

$$3 \times 670^2 = 1346700$$

as before. Or note that the sum of the first n odd integers is n^2.

Answer is (B).

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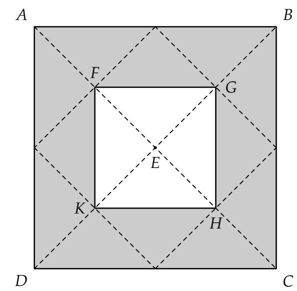
10. If a and b are the side lengths of the rectangle then we have $a \cdot b = 24$ and we want to know the possible values of $2(a + b)$ given that a and b are integers. By re-labeling if necessary, we see the possible integer values of a are 1, 2, 3, and 4, and these lead to the respective perimeter values 50, 28, 22, and 20. Evidently 24 metres is not a possible value for the perimeter.

Answer is (D).

11. Dropping perpendiculars from the midpoints of the sides of the larger square to the given midpoints on the smaller square partitions the larger square into 16 congruent triangles. Since the shaded region consists of 12 of these triangles, the ratio of the unshaded area to the shaded area is 4 : 12 which simplifies to 1 : 3.



Alternative solution

Obviously, $FGHK$ is a square. Further, triangles ABE and FGE are similar and $\overline{FE} = \frac{1}{2}\overline{AE}$. Hence, $\overline{FG} = \frac{1}{2}\overline{AB}$. So the area of the square $FGHK$ is $\frac{1}{4}$ of the area of the square $ABCD$. Hence, the ratio of the unshaded area to the shaded area is 1 : 3.

Answer is (C).

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12. Jerry's statement "I lie on Sundays" means that Jerry is lying since both tell the truth on Sunday. Jerry's statement "I lie on Saturdays" is a lie, since Jerry is lying today. So the day must be Monday, Tuesday or Wednesday. Since these are days on which Kelly tells the truth, Kelly's statement "I will lie tomorrow" means that the next day is a day on which Kelly lies that immediately follows a day when Kelly tells the truth. Therefore, it must be Wednesday.

Answer is (B).

Senior Preliminary

1. If the amount that Mary Lou earns is equivalent to working 12 hours at her usual pay then $M + 1.5N = 12$. For the total number of hours worked to be as large as possible, the number of overtime hours must be as small as possible. Further, N must be even if $M + 1.5N$ is to be an integer. So take $N = 2$, which gives $M = 9$, for a total of 11 hours worked.

Answer is (A).

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2. Add pairs of numbers in the given expression, beginning with the outermost pair and working inwards. Since the sum of each pair is 1 and there are evidently 50 such pairs, the value of the expression is 50.

Alternative solution

Write the sum as

$$\begin{aligned}100 - 1 + 98 - 3 + 96 - 5 + \cdots \\ +4 - 97 + 2 - 99 &= (100 + 98 + \cdots + 2) - (1 + 3 + \cdots + 99) \\ &= 2\underbrace{(50 + 49 + \cdots + 1)}_{50\text{ terms}} - \underbrace{(1 + 3 + \cdots + 99)}_{50\text{ terms}} \\ &= 2\left[\left(\frac{1}{2}\right)50\,(51)\right] - 50^2 = 50 \times 51 - 50^2 = 50\end{aligned}$$

Answer is (B).

3. If v is the volume of the water and ℓ is the level of the water when the box is placed on one of its 7 by 18 faces, then, since the volume of water remains fixed, $v = 15 \cdot 7 \cdot 12 = 7 \cdot 18 \cdot \ell$, and hence $\ell = 10$.

Answer is (B).

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4. If n, V, and $\overline{v}$ denote respectively the number of average raindrops, the volume of the rain on the city, and the average volume of a raindrop, then

$$\begin{aligned} n = \frac{V}{\overline{v}} &= \frac{(10\,\text{km})^2 \cdot 1\,\text{cm}}{10\,\text{mm}^3} = \frac{\left(10 \times 10^5\,\text{cm}\right)^2 \cdot 1\,\text{cm}}{10\ \left(10^{-1}\text{cm}\right)^3} \\ &= \frac{100 \times 10^{10}\,\text{cm}^3}{10\ \left(10^{-1}\text{cm}\right)^3} = \frac{100 \times 10^{10}\,\text{cm}^3}{10 \times 10^{-3}\text{cm}^3} = 10^{14} \end{aligned}$$

Answer is (E).

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5. If N is odd, then, since Tim receives his gift in increments of 2 and it must exceed N, he receives $N+1$ dollars. Similarly, since Ursula receives her gift in increments of 5 and it must exceed Tim's amount of $N+1$, Ursula receives no more than $N+6$ dollars. Reasoning in the same way if N is even, we see Tim must receive $N+2$ dollars and then Ursula receives no more than $N+7$ dollars.

Answer is (D).

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6. If t, ℓ, and b denote, respectively, my height this year, last year, and the year before that, then $t = \ell + \frac{1}{10}\ell$ and $\ell = b + \frac{1}{5}b$, so

$$t = b + \tfrac{1}{5}b + \tfrac{1}{10}\left(b + \tfrac{1}{5}b\right) = b + \tfrac{32}{100}b$$

Therefore the percentage increase over two years is 32.

Answer is (E).

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7. Note that $4y^2$ is even for any integer value of y, so x^2, and hence x, must be even for any integer value of y. Further, x cannot be larger than 10. So the only possible non-negative values of x are $x = 0$, $x = 2$, $x = 4$, $x = 6$, $x = 8$, and $x = 10$. Only $x = 0$, $x = 6$, $x = 8$, and $x = 10$ give integer values for y. Plugging these x values into the equation gives:

x	y	number of points
0	± 5	2
± 6	± 4	4
± 8	± 3	4
± 10	0	2

Giving a total of 12 points.

Answer is (D).

8. The sum of all the positive integers less than 2010 is

$$\frac{2009 \times 2010}{2} = 3\left(\frac{2009 \times 670}{2}\right)$$

The sum of all the positive integers less than 2010 which are divisible by three is

$$3 + 6 + 9 + \cdots + 2007 = 3\,(1 + 2 + 3 + \cdots 669) = 3\left(\frac{669 \times 670}{2}\right)$$

The required sum is

$$3 \times 670\left(\frac{2009 - 669}{2}\right) = 3 \times 670\left(\frac{1340}{2}\right) = 3 \times 670^2 = 1346700$$

Alternative solution
The sum of the integers less than 2010 that are not divisible by 3 is

$$(1+2) + (4+5) + (7+8) + \cdots + (2008 + 2009) = 3 + 9 + 15 + \cdots + 4017 = 3\,(1 + 3 + 5 + \cdots + 1339)$$

This is an arithmetic progression of 670 terms with starting value 1 and difference 2. The sum is

$$670 \times \frac{1 + 1339}{2} = 670^2$$

giving the value of the sum as

$$3 \times 670^2 = 1346700$$

as before. Or note that the sum of the first n odd integers is n^2.

Answer is (B).

9. The centre of the circle follows a smaller hexagon, as shown. Triangles OPQ and OAB are both 30°-60°-90° and $\overline{OQ} = 4$ and $\overline{AP} = 1$. Hence

$$\overline{OP} = 4\left(\frac{\sqrt{3}}{2}\right) = 2\sqrt{3} \quad \text{and} \quad \overline{OA} = 2\sqrt{3} - 1$$

Triangles OPQ and OAB are similar, so that

$$\frac{\overline{OB}}{2\sqrt{3}-1} = \frac{4}{2\sqrt{3}} \Rightarrow \overline{OB} = \frac{2}{\sqrt{3}}\left(2\sqrt{3}-1\right) = 4 - \frac{2\sqrt{3}}{3}$$

This distance is also the length of the sides of the inner hexagon. Hence, the total distance traveled by the centre of the circle is

$$6\left(4 - \tfrac{2}{3}\sqrt{3}\right) = 24 - 4\sqrt{3}$$

Answer is (E).

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10. Based on the coin flip, on average, you will leave 3 of the 6 questions blank for a total of 3 points. For each of the remaining 3 questions you have a one-third chance of getting the question correct, since you know that two of the answers are incorrect. So for these 3 questions, on average, you will get one correct for 5 points and the other two incorrect for no points. Therefore, your average score for the six questions is 8.

Answer is (D).

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11. Jerry's statement "I lie on Sundays" means that Jerry is lying, since both tell the truth on Sundays. Jerry's statement "I lie on Saturdays" is a lie, since Jerry is lying today. So the day must be Monday, Tuesday or Wednesday. Since these are days on which Kelly tells the truth, Kelly's statement "I will lie tomorrow" means that the next day is a day on which Kelly lies that immediately follows a day when Kelly tells the truth. Therefore, it must be Wednesday.

Answer is (B).

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12. First note that the points $(0,0)$, $(1,2)$, and $(2,4)$ are collinear, so they cannot all lie on the same circle. Hence, there can be no more than 6 of the points that lie on the same circle. Next, if the points $(3,1)$ and $(3,3)$, or $(2,0)$ and $(2,4)$, are on the same circle, then the centre must be at the point $(x,2)$ for some x. If all four of these points lie on the same circle, then x must satisfy the equation

$$(3-x)^2 + (3-2)^2 = (2-x)^2 + (4-2)^2 \Rightarrow 9 - 6x + x^2 + 1 = 4 - 4x + x^2 + 4$$

Solving for x gives $x = 1$. So $(1,2)$ is the centre of the circle. Since $(1,2)$ is the midpoint of the line segment joining $(2,4)$ and $(0,0)$, the point $(0,0)$ lies on the same circle. Finally, note that the distance of the point $(4,3)$ is greater than the distance of any of the other points from $(1,2)$. Hence, at least 5 of the points can lie on the same circle. Finally, the only way that 6 points can lie on the same circle, since only two of $(0,0)$, $(2,0)$, and $(2,4)$ can, is if all of the rest of the points lie on the circle. But this requires that $(3,1)$, $(3,3)$, $(2,0)$, and $(2,4)$ lie on the circle, so that $(1,2)$ is the centre of the circle. But, then, $(4,3)$ is not on the circle. Therefore, the largest number of the points that can lie on the same circle is 5.

Answer is (C).

Junior Final, Part A

1. Let a_n be the value obtained after n repetitions. Then $a_0 = 3$ and

$$a_n = 2a_{n-1} - 1,\ n \geq 1$$

Calculating the first few numbers in the sequence gives:

$$\begin{aligned} a_0 &= 3 \\ a_1 &= 2 \times 3 - 1 = 5 \\ a_2 &= 2 \times 5 - 1 = 9 \\ a_3 &= 2 \times 9 - 1 = 17 \end{aligned}$$

It appears (from this short list) that $a_n = 2^{n+1} + 1$. In particular, the required value is $a_{30} = 2^{31} + 1$. Note: To prove the formula above by induction assume that $a_k = 2^{k+1} + 1$, and calculate

$$a_{k+1} = 2 \times a_k - 1 = 2 \times (2^{k+1} + 1) - 1 = 2^{k+2} + 2 - 1 = 2^{k+2} + 1$$

This is what the proposed formula says it should be with $n = k + 1$. Hence, $a_n = 2^{n+1} + 1$ for all n.

Answer is (A).

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2. Observe that 2^n has only one prime factor, namely 2. Since one of m and $m + 1$ is odd and the other is even, $m\,(m + 1)$ will have a factor of 2. For this to be the only prime factor, the other factor must be 1. So the only possible value for m is $m = 1$, giving $n = 1$, since then $m\,(m + 1) = 2 = 2^1$.

Answer is (B).

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3. Given that $150 = 2 \times 3 \times 5^2$, listing all possible boxes with the shortest dimensions in increasing order and the corresponding surface area gives:

$$\begin{aligned} 1 \times 1 \times 150&: \quad \text{SA} = 4 \times 150 + 2 \times 1 = 602 \\ 1 \times 2 \times 75&: \quad \text{SA} = 2 \times 150 + 2 \times 75 + 2 \times 2 = 454 \\ 1 \times 3 \times 50&: \quad \text{SA} = 2 \times 150 + 2 \times 50 + 2 \times 3 = 406 \\ 1 \times 5 \times 30&: \quad \text{SA} = 2 \times 150 + 2 \times 30 + 2 \times 5 = 370 \\ 1 \times 6 \times 25&: \quad \text{SA} = 2 \times 150 + 2 \times 25 + 2 \times 6 = 362 \\ 1 \times 10 \times 15&: \quad \text{SA} = 2 \times 150 + 2 \times 15 + 2 \times 10 = 350 \\ 2 \times 3 \times 25&: \quad \text{SA} = 2 \times 75 + 2 \times 50 + 2 \times 6 = 262 \\ 2 \times 5 \times 15&: \quad \text{SA} = 2 \times 75 + 2 \times 30 + 2 \times 10 = 230 \\ 3 \times 5 \times 10&: \quad \text{SA} = 2 \times 50 + 2 \times 30 + 2 \times 15 = 190 \\ 5 \times 5 \times 6&: \quad \text{SA} = 4 \times 30 + 2 \times 25 = 170 \end{aligned}$$

The minimum possible surface area is 170 cubic units.

Alternative solution

The box with the minimum surface area is the one which is "closest to" a sphere. In this case it is the $5 \times 5 \times 6$ box which has an area of

$$2 \times 5 \times 5 + 4 \times 5 \times 6 = 170$$

Answer is (C).

4. Since 4 digits = 1 cm, then 1 digit = 0.25 cm and

$$13 \times 10^6 \text{ digits} = 3.25 \times 10^6 \text{ cm} = 3.25 \times 10^4 \text{ m} = 3.25 \times 10 \text{ km} = 32.5 \text{ km}$$

Answer is (E).

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5. Observe that $40 = 49 - 9 = 7^2 - 3^2$ and $44 = 144 - 100 = 12^2 - 10^2$. Further, any odd integer can be expressed as a difference of squares, since $2n + 1 = (n+1)^2 - n^2$. Hence, both 41 and 43 can be expressed as a difference of squares. Therefore, 42 is the only number of the given numbers that cannot be expressed as a difference of squares.

Alternative solution

If the number x can be written as a difference of squares, then $x = a^2 - b^2 = (a+b)(a-b)$. Letting $a + b = m$ and $a - b = n$, means that $x = m \times n$ and $a = \dfrac{m+n}{2}$ and $b = \dfrac{m-n}{2}$. For a and b to be integers m and n must have the same parity (both even or both odd). This requires that x is either odd or divisible by 4. Hence, 42 cannot be expressed as a difference of squares. Considering the other four distracters:

- For $40 = 2^3 \times 5$ take $m = 2$ and $n = 20$ giving $40 = 11^2 - 9^2$, or $m = 4$ and $n = 10$ giving $40 = 7^2 - 3^2$.
- For 41 and 43, which are odd, take $m = 1$ giving $41 = 21^2 - 20^2$ and $43 = 22^2 - 21^2$.
- For $44 = 2^2 \times 11$ take $m = 2$ and $n = 22$ giving $44 = 12^2 - 10^2$.

So 42 is the only number of the five given numbers that cannot be written as the difference of the squares of two integers in at least one way.

Answer is (C).

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6. Count the number of possibilities based on the number of quarters used, as in the table below.

One quarter		Two quarters		Three quarters	
Dimes	Nickels	Dimes	Nickels	Dimes	Nickels
1	13	1	8	1	3
2	11	2	6	2	1
3	9	3	4		
4	7	4	2		
5	5				
6	3				
7	1				

From the table above, there are 13 ways to make change.

Answer is (D).

7. Let the letters to the students be represented by 1, 2, 3, and 4. Indicate which letter went in which envelope by an ordered 4-tuple; for example, $(1, 2, 3, 4)$ indicates that each letter was put in the correct envelope. There are three ways the second letter can be put in the first envelope and have none of the other three in their correct envelopes: $(2, 1, 4, 3)$, $(2, 4, 1, 3)$, and $(2, 3, 4, 1)$. Similarly, there must be three ways in which the third letter gets put in the first envelope and three ways in which the fourth letter gets put in the first envelope. This gives a total of 9 ways.

   Alternative solution

   There are $4! = 24$ permutations of four symbols. It is desired to count those permutations in which no symbol is mapped to itself. Do this by first counting those permutations in which at least one symbol is mapped to itself. There is 1 permutation in which all four symbols are mapped to themselves. If three symbols map to themselves, then the fourth is also mapped to itself. So there are no permutations in which exactly three symbols map to themselves. There are 6 ways to choose two of the symbols to map to themselves, and, once these are chosen, the other two can only be mapped in one way. So there are 6 permutations in which exactly two symbols map to themselves. There are 4 ways to choose one symbol to map to itself. Once this symbol is chosen the other three can be mapped in two ways so that none of them map to themselves. So, there are 8 permutations in which one symbol maps to itself. This gives a total of $1 + 6 + 8 = 15$ permutations in which at least one symbol maps to itself. Hence, there are $24 - 15 = 9$ permutations in which no symbol maps to itself.

   Alternative solution

   A permutation of n symbols in which no symbol is mapped to itself is a *derangement*. Let $D(n)$ be the number of derangements of n symbols. Obviously, $D(1) = 0$ and $D(2) = 1$. After that $D(n)$ can be found recursively. There are $n - 1$ ways to select the symbol to which first symbol maps. Suppose that this is the second symbol. Then there are two possibilities: either the second symbol is mapped to the first symbol or to a different symbol. In the first case, there are $n - 2$ symbols left to derange, for a total of $D(n-2)$ possibilities. In the second case, since the second symbol cannot map to the first symbol (think of it as the new first symbol), there are $n - 1$ symbols to derange, for a total of $D(n-1)$ possibilities. Hence, the number of derangements of n symbols satisfies the recursion equation

   $$D(n) = (n-1)\left[D(n-1) + D(n-2)\right]$$

   Using this equation gives $D(3) = 2(D(2) + D(1)) = 2$ and $D(4) = 3(D(3) + D(2)) = 9$.

   Answer is (C).

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8. Let a be the side length of the outer square. Since the vertices of the middle square divide each side of the outer square into segments whose lengths are in ratio 3 : 1, the length of the longer segment of each side of the outer square is $\frac{3}{4}a$ and length of the shorter segment is $\frac{1}{4}a$. Hence, by Pythagoras' theorem, the side length of the middle square is $\frac{1}{4}\sqrt{10}a$. By similarity, the side length of the inner square is

   $$\left(\tfrac{1}{4}\sqrt{10}\right)\left(\tfrac{1}{4}\sqrt{10}\right)a = \tfrac{10}{16}a = \tfrac{5}{8}a$$

   Therefore, the ratio of the area of the inner square to the area of the outer square is

   $$\left(\tfrac{5}{8}a\right)^2 : a^2 = 25 : 64$$

   Answer is (D).

9. Writing $\frac{97}{19}$ in the required form gives

$$\frac{97}{19} = 5 + \frac{2}{19} = 5 + \frac{1}{\left(\frac{19}{2}\right)} = 5 + \frac{1}{9 + \frac{1}{2}}$$

Hence, $w = 5$, $x = 9$, and $y = 2$ giving $w + x + y = 16$.

Alternative solution

First observe that for positive integers x and y

$$\frac{1}{1 + \frac{1}{y}} < 1$$

then writing

$$\frac{97}{19} = 5 + \frac{2}{19}$$

shows that $w = 5$. Hence

$$\frac{1}{x + \frac{1}{y}} = \frac{2}{19} \Rightarrow y\,(19 - 2x) = 2$$

The only positive integers that satisfy this equation are $y = 2$ and $x = 9$. Hence, $x + y + w = 16$.

Answer is (B).

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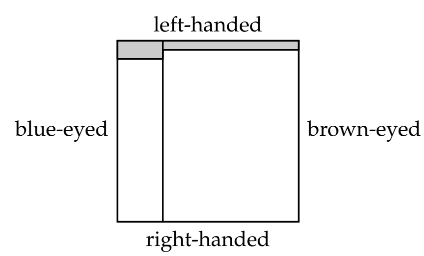
10. In the diagram the square represents all of the students and the vertical line through the square divides the students into the 25% blue-eyed students and 75% brown-eyed students. The two horizontal lines divide the blue and brown-eyed groups into the 10% left-handed and 5% left-handed groups, respectively. The proportion of the students who are blue-eyed and left-handed, as represented by the area of the rectangle in the upper left, is $0.25 \times 0.1 = 0.025$ or 2.5%. The proportion of the students who are brown-eyed and left-handed, as represented by the area of the rectangle in the upper right, is $0.75 \times 0.05 = 0.0375$ or 3.75%. The proportion of the students who are left-handed is the sum of these two areas, $0.025 + 0.0375 = 0.0625$. This is the shaded portion of the diagram. The proportion of the left-handed students who are blue-eyed is $\frac{0.025}{0.0625} = 0.40 = 40\%$.



Alternative solution

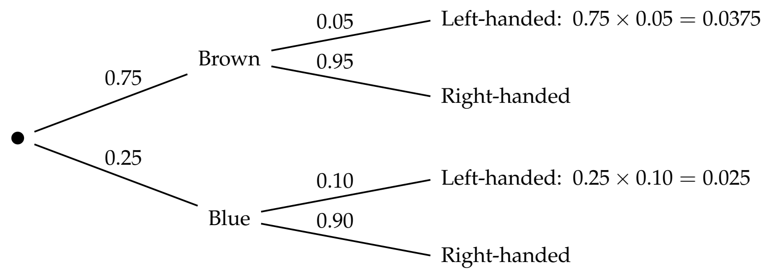
Consider a group of 80 students. From this group $0.75 \times 80 = 60$ have brown eyes and $0.25 \times 80 = 20$ have blue eyes. Of the 60 brown-eyed students $0.05 \times 60 = 3$ are left-handed, and of the 20 blue-eyed students $0.10 \times 20 = 2$ are left-handed. Hence, there are 5 left-handed students altogether and 2 of them are blue-eyed. Therefore, the proportion of left-handed students who are blue-eyed is $\frac{2}{5} = 0.40 = 40\%$.

Problem 10 continued . . .

. . . Problem 10 continued

Alternative solution

Representing the given information in a tree as shown



shows that the proportion of students who are left-handed and brown-eyed is $0.75 \times 0.05 = 0.0375$ and the proportion who are left-handed and blue-eyed is $0.25 \times 0.10 = 0.025$. So the proportion of the left-handed students is $0.0375 + 0.025 = 0.0625$. So the proportion of left-handed students who are blue-eyed is $\dfrac{0.025}{0.0625} = 0.40 = 40\%$.

Answer is (E).

Junior Final, Part B

1. (a) The sum of all the numbers is

$$3 + 8 + 13 + \cdots + 1998 + 2003 + 2008$$
$$= 3 + (3+5) + (3 + 2 \times 5) + \cdots + (3 + 399 \times 5) + (3 + 400 \times 5) + (3 + 401 \times 5)$$

which is an arithmetic series with a common difference of 5 and 402 terms, since $\dfrac{2008-3}{2} = 401$. Hence, the value of the sum is

$$\left(\frac{402}{2}\right)(3 + 2008) = 201 \times 2011 = 404211$$

OR, using summation notation and the standard sum $\displaystyle\sum_{i=1}^{n} i = \frac{n\,(n+1)}{2}$ gives

$$\begin{aligned} &3 + 8 + 13 + 18 + \cdots + 1993 + 1998 + 2003 + 2008 \\ &= \sum_{i=0}^{401} (5i+3) = 5\sum_{i=1}^{401} i + 3\sum_{i=0}^{401} 1 \\ &= 5 \times \frac{401 \times 402}{2} + 3 \times 402 = 5 \times 401 \times 201 + 3 \times 2 \times 201 \\ &= 201\,(5 \times 401 + 6) = 201 \times 2011 = 404211 \end{aligned}$$

Problem 1 continued . . .

...Problem 1 continued

Alternative solution

The sum of the numbers for which the units digit is '3' is

$$3 + 13 + \cdots + 1993 + 2003 = 3 + (3 + 10) + \cdots + (3 + 199 \times 10) + (3 + 200 \times 5)$$

The sum of the numbers for which the units digit is '3' is

$$8 + 18 + \cdots + 1998 + 2008 = 8 + (8 + 10) + \cdots + (8 + 199 \times 10) + (8 + 200 \times 5)$$

Both are arithmetic series with a common difference of 10 and 201 terms. The first and last terms are 3 and 2003 for the first, and 8 and 2008 for the second. Adding the two together gives

$$\left(\frac{201}{2}\right)(3 + 2003) + \left(\frac{201}{2}\right)(8 + 2008) = 201\,(1003 + 1008) = 201 \times 2011 = 404211$$

Thus, the sum is 404211.

Answer: The sum is 404211.

(b) Let x and y be the number of units of water in can X and Y, respectively, at the start. After the first step, can X contains $x - y$ units and can Y contains $2y$ units. After the second step can X contains $2(x - y) = 2x - 2y$ units and can Y contains $2y - (x - y) = 3y - x$ units. After the last step can X contains $2x - 2y - (3y - x) = 3x - 5y$ units and can Y contains $2(3y - x) = 6y - 2x$ units. At this stage

$$3x - 5y = 24 \text{ and } 6y - 2x = 24$$

Adding the two equations gives $x + y = 48$ and dividing the second equation by 2 gives $3y - x = 12$. Adding these two equations gives $4y = 60$. Hence, $y = 15$ and $x = 48 - 15 = 33$. So at the start can X contained 33 units and Y contained 15 units.

Alternative solution

Observe that each time the can that is gaining water actually doubles the amount of water from the previous its previous amount. Hence, the problem can be solved by working backwards. After the water is poured from X into Y at the final step, Y contains 24 units. Hence, it had 12 units before that step and then X had $24 + 12 = 36$ units. Prior to that, the amount of water poured from Y into X was $\frac{1}{2} \times 36 = 18$. Using this logic gives the table below

	X	Y
After final step	24	24
After previous step	36	12
After previuos step	18	30
Starting point	33	15

Therefore, at the start can X contained 33 units and Y contained 15 units.

Answer: X contained 33 units and Y contained 15 units at the start.

2. Let h be the height of triangle APE and $\ell = \overline{AE}$. Then

$$\frac{1}{2}\ell h = 12$$

and $\overline{AB} = \overline{BC} = \overline{CD} = \overline{DE} = \frac{1}{4}\ell$. The area of triangles APB, BPC, CPD, and DPE is

$$A_1 = \frac{1}{2}\left(\frac{1}{4}\ell h\right)$$

The area of triangles APC, BPD, and CPE is

$$A_2 = \frac{1}{2}\left(\frac{1}{2}\ell h\right)$$

The area of triangles APD and BPE is

$$A_3 = \frac{1}{2}\left(\frac{3}{4}\ell h\right)$$

So the sum of the areas of all of the triangles, including triangle APE, is:

$$\frac{1}{2}\ell h + 4A_1 + 3A_2 + 2A_3 = \frac{1}{2}\ell h\left(1 + 1 + \frac{3}{2} + \frac{3}{2}\right) = 12(5) = 60$$

Alternative solution

Since all of the triangles have the same height, any pair of triangles with the same base have equal area. Therefore, there are

- four triangles with area 3: ABP, BCP, CDP, DEP
- three triangles with area 6: ACP, BDP, CEP
- two triangles with area 9: ADP, BEP
- one triangle with area 12: AEP

So the total area is

$$A = 4 \times 3 + 3 \times 6 + 2 \times 9 + 1 \times 12 = 60$$

3. Since ABS is an equilateral triangle $\overline{AS} = \overline{BS}$, and, since $PQRS$ is a square $\overline{PS} = \overline{RS}$. Hence, triangles APS and BRS are congruent. Thus, $\overline{AP} = \overline{BR}$, so that $\overline{AQ} = \overline{BQ}$. Therefore, ABQ is an isosceles right triangle. Without loss of generality, let the side length of the equilateral triangle $\overline{AB} = \overline{AS} = \overline{BS} = 2$. Then, the height of ABS is $\sqrt{3}$, and $\overline{BQ} = \overline{AQ} = \sqrt{2}$. Let a be the side length of the square. Then, $\overline{AP} = \overline{BR} = a - \sqrt{2}$. The total area of the square is

$$\begin{aligned} a^2 &= \text{Area } ABS + \text{Area } APS + \text{Area } BRS + \text{Area } ABQ \\ &= \sqrt{3} + 2\left[\tfrac{1}{2}a\left(a - \sqrt{2}\right)\right] + 1 = \sqrt{3} + 1 + a^2 - a\sqrt{2} \end{aligned}$$

Solving for a gives

$$a\sqrt{2} = \sqrt{3} + 1 \Rightarrow a = \frac{\sqrt{3}+1}{\sqrt{2}}$$

The area of APS is

$$\begin{aligned} \frac{1}{2}a\left(a - \sqrt{2}\right) &= \frac{1}{2}\left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}+1}{\sqrt{2}} - \sqrt{2}\right) \\ &= \frac{1}{2}\left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}-1}{\sqrt{2}}\right) = \frac{1}{2}\left(\frac{3-1}{2}\right) = \frac{1}{2} \end{aligned}$$

But the area of triangle ABQ is $\frac{1}{2}\left(\sqrt{2}\right)\left(\sqrt{2}\right) = 1$. So the ratio of the area of triangle APS to the area of triangle ABQ is 1 : 2.

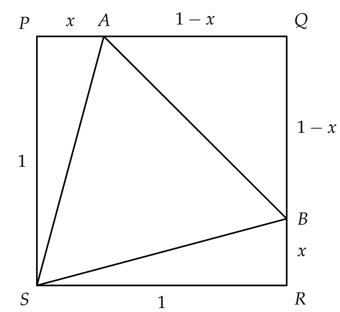
Alternative solution

Let the side length of the square be 1, and let $x = \overline{AP} = \overline{BR}$. Then $\overline{AQ} = \overline{BQ} = 1 - x$. Since $\overline{AS} = \overline{AB}$ using Pythagoras' theorem on triangles APS and ABQ gives

$$1 + x^2 = 2(1-x)^2 \Rightarrow 1 + x^2 = 2 - 4x + 2x^2 \Rightarrow x^2 - 4x + 1 = 0$$

Solving for x gives

$$x = \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}$$



Labeled for first Alternative solution

Since $0 \le x \le 1$, we take the negative sign to give $x = 2 - \sqrt{3}$. So that $1 - x = \sqrt{3} - 1$. Then the area of triangle APS is

$$\text{Area } APS = \frac{1}{2}(1)\left(2 - \sqrt{3}\right) = \frac{2-\sqrt{3}}{2}$$

and the area of triangle ABQ is

$$\text{Area } ABQ = \frac{1}{2}\left(\sqrt{3}-1\right)^2 = \frac{1}{2}\left(4 - 2\sqrt{3}\right) = 2 - \sqrt{3}$$

So the ratio of the areas of the triangles APS and ABQ is 1 : 2.

Problem 3 continued ...

...Problem 3 continued

Note

It is not necessary to solve for x in the quadratic equation above. The area of triangle APS is

$$\text{Area } APS = \frac{x}{2}$$

The area of triangle ABQ is

$$\text{Area } ABQ = \frac{(1-x)^2}{2} = \frac{1+x^2}{4} = x$$

where the second form comes from the equality giving the quadratic equation and the last form comes from solving the final form of the quadratic equation for $1 + x^2$. Hence, the ratio of the areas is $\frac{1}{2}$, as above.

Alternative solution (using trigonometry)

Since triangles APS and RSB are congruent, $\angle PSA = \angle SRB = 15°$. Let $\overline{AB} = a$, so $\overline{AQ} = a/\sqrt{2}$, $\overline{AP} = a\cos 15°$, and $\overline{PS} = a \sin 15°$. Then

$$\text{Area } \Delta ABQ = \frac{1}{2}\frac{a}{\sqrt{2}}\frac{a}{\sqrt{2}} = \frac{a^2}{4}$$

$$\text{Area } \Delta APS = \frac{1}{2} a \sin 15° a \cos 15° = \frac{a^2}{4}\sin 30° = \frac{a^2}{8}$$

where we have used $\sin 2\theta = 2 \sin\theta \cos\theta$. Then

$$\frac{\text{Area } \Delta APS}{\text{Area } \Delta ABQ} = \frac{a^2/8}{a^2/4} = \frac{1}{2}$$

Answer: The ratio of the areas is $1 : 2$**.**

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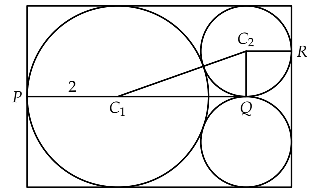
4. Note that there are exactly 10 distinct sums of distinct pairs from a set of 5 numbers, so every possible pair will be used to generate one of the ten numbers. Since the set of sums contains zero there must be one pair of the form $-n$ and n. Further, there can only be one negative number and zero cannot be used, otherwise at least one of the sums will be negative. Taking $n = 1$ means that -1 and 1 are included. Then the set must contain 2 and 3 to produce 1 and 2. But then 3 is one of the sums, so all of the required sums cannot be generated. Taking $n = 2$ means that -2 and 2 are included. Then the set must include 3 to produce 1. But then 5 is one of the sums, so all of the required sums cannot be generated. For -3 and 3, the numbers 4, 5, and 7 must be included in order to generate 1, 2, and 4. Then, $8 = 3 + 5$, $9 = 4 + 5$, $10 = 3 + 7$, $11 = 4 + 7$, and $12 = 5 + 7$. Therefore, the set of five distinct integers that gives the required pairwise sums is $\{-3, 3, 4, 5, 7\}$.

Alternative solution

Let the five distinct integers be represented by a, b, c, d, and e where $a < b < c < d < e$. Since each of a, b, c, d, and e appear in four pairwise sums and $0 + 1 + 2 + 4 + 7 + 8 + 9 + 10 + 11 + 12 = 64$, it follows that $a + b + c + d + e = 64/4 = 16$. Noting that the largest and smallest sums, respectively, are $d + e = 12$ and $a + b = 0$ gives $c = 16 - (12 + 0) = 4$. Observing that the second largest sum is $c + e = 11$ gives $e = 7$. Then $d + e = 12$ gives $d = 5$. Further, $b + e = 10$ giving $b = 3$. Finally, $a + b = 0$ gives $a = -b = -3$. The integers are -3, 3, 4, 5, and 7 in increasing order.

Answer: The numbers are $-3, 3, 4, 5, 7$**.**

5. Let C_1 denote the centre of the larger circle and C_2 the centre of the upper smaller circle. Let P be the intersection of the larger circle and the left vertical side of the rectangle, as shown. Let R be the intersection point of the upper smaller circle and the right vertical side of the rectangle, as shown. Finally, let Q denote the intersection point of the two smaller circles. Then, the width of the rectangle is equal to



$$\overline{PC_1} + \overline{C_1Q} + \overline{C_2R}$$

Note that the radius of the larger circle is 2 since the circle is tangent to the sides of the rectangle. For a similar reason, the radii of the smaller circles are 1. It follows that:

$$\overline{PC_1} = 2 \text{ and } \overline{C_2R} = 1$$

It remains to find the length $\overline{C_1Q}$. Since the triangle C_1QC_2 is a right-angle triangle with hypotenuse equal to the sum of the radii of the large circle and one of the small circles, by Pythagoras' Theorem

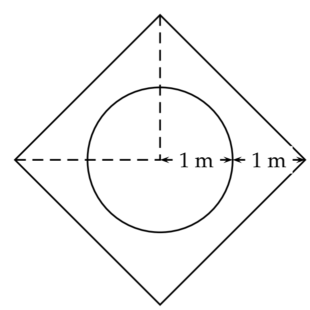
$$\overline{C_1Q} = \sqrt{(1+2)^2 - 1^2} = 2\sqrt{2}$$

Therefore, the width of the rectangle is $2 + 1 + 2\sqrt{2} = 3 + 2\sqrt{2}$.

Answer: The width of the rectangle is $3 + 2\sqrt{2}$.

Senior Final, Part A

1. The side of the tablecloth is the hypotenuse of an isosceles right triangle with side 2 m so it is $2\sqrt{2}$ m.



Answer is (A).

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2. Recognizing the expression as

$$2009^2 + 2 \times 2009 + 1 = (2009 + 1)^2 = 2010^2$$

it is only necessary to find the prime factors of 2010. But $2010 = 201 \times 10 = 3 \times 67 \times 2 \times 5$. Since 67 is prime, there are 4 prime factors.

Answer is (C).

---

3. See the solution to Junior Final, Part A, Problem 3.

Answer is (C).

4. See the solution to Junior Final, Part B, Problem 2.

Answer is (E).

---

5. The sum of the positive integers less than 1000 that are divisible by 9 is

$$9 + 18 + 27 + \cdots + 999 = 9\,(1 + 2 + 3 \cdots + 111) = 9\left(\frac{111 \times 112}{2}\right) = 55944$$

The sum of the positive integers less than 1000 that are divisible by 11 is

$$11 + 22 + 33 + \cdots + 990 = 11(1 + 2 + 3 \cdots + 90) = 11\left(\frac{90 \times 91}{2}\right) = 45045$$

The sum of the positive integers less than 1000 that are divisible by 9 and 11, that is divisible by 99, is

$$99 + 198 + 297 + \cdots + 990 = 99\,(1 + 2 + 3 \cdots + 10) = 99\left(\frac{10 \times 11}{2}\right) = 5445$$

This sum is included in both of the previous two, so the sum of all positive integers less than 1000 that are divisible by 9 or 11, but not both is

$$(55944 - 5445) + (45045 - 5445) = 55944 + 45045 - 2 \times 5445 = 90099$$

Answer is (A).

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6. The height of the stack is

$$12\left(1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \cdots + \left(\frac{2}{3}\right)^5\right) = 12\left(\frac{1 - \left(\frac{2}{3}\right)^6}{1 - \frac{2}{3}}\right) = 36\left(1 - \frac{2^6}{3^6}\right)$$
$$= 36 - \frac{2^8}{3^4} = 36 - \frac{256}{81} \approx 36 - 3.16 \approx 32.84$$

Without using decimals

$$36 - \frac{2^8}{3^4} = 36 - 16\left(\frac{16}{81}\right) \approx 36 - 16\left(\frac{1}{5}\right) \approx 36 - \frac{16}{5} \approx 36 - 3 = 33$$

Answer is (D).

---

7. The measure of each interior angle of a regular polygon with n sides, in degrees, is $180 - \dfrac{360}{n}$. So the sum of all of the angles is $180n - 360$. It is desired to find n so that

$$n^2 > 180n - 360 \Rightarrow n^2 - 180n + 360 > 0$$

This inequality is satisfied for $n = 1$ and $n = 2$, but neither of these values gives a polygon. For $n = 3$ the inequality is not true. Hence, a solution with $n > 3$ is required. First solve the equation

$$n^2 - 180n + 360 = 0 \Rightarrow n = 6\left(15 \pm \sqrt{215}\right)$$

Taking the plus sign to give a value of $n > 3$ gives

$$n = 6\left(15 + \sqrt{215}\right) > 6(15 + 14) = 174$$

So, $n \geq 160$.

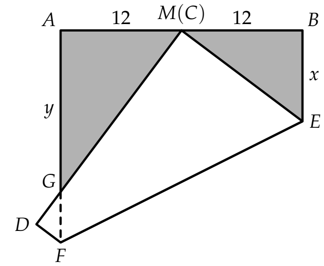
Answer is (E).

8. Let $x = \overline{BE}$ and $y = \overline{AG}$. Triangles BEC and ACG are similar, so that

$$\frac{x}{12} = \frac{12}{y} \Rightarrow xy = 144$$

Further, $\overline{BE} + \overline{CE} = 24$, since BE and CE make up one side of the sheet of paper, and, using Pythagoras' theorem

$$\overline{CE} = \sqrt{12^2 + x^2}$$



Hence

$$x + \sqrt{12^2 + x^2} = 24 \Rightarrow 12^2 + x^2 = (24 - x)^2 = 24^2 - 48x + x^2$$

Then, solving for x gives

$$48x = 432 \Rightarrow x = 9$$

and, finally, solving for y gives

$$9y = 144 \Rightarrow y = 16$$

Answer is (B).

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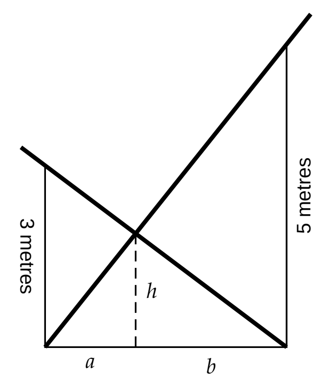
9. Let h be the distance from the bottom of the trench to the point of intersection of the ladders, a the distance from the corner on the short side to the point on the bottom of the trench just below the point of intersection of the ladders, and b the corresponding distance to the corner on the high side of the trench. (See the diagram.) Then by similar triangles

$$\frac{h}{3} = \frac{b}{a+b} \quad \text{and} \quad \frac{h}{5} = \frac{a}{a+b}$$



hence

$$h\,(a+b) = 3b = 5a$$

so that

$$h = \frac{3b}{a+b} = \frac{15b}{5a+5b} = \frac{15b}{3b+5b} = \frac{15b}{8b} = \frac{15}{8}$$

Answer is (D).

---

10. We need to know when the two hands of the clock are separated by exactly 180°. If m is the number of minutes after 20:10, the number of degrees clockwise that the minute hand is from 12:00 (vertical) is $60 + 6m$, since the minute hand moves at 6 degrees per minute. At this time the number of degrees clockwise that the hour hand is from 12:00 is $245 + \frac{1}{2}m$, since the hour hands moves one twelfth as fast as the minute hand, that is, at one-half a degree per minute. When the hour and minute hands form a straight line

$$60 + 6m + 180 = 245 + \tfrac{1}{2}m$$

Solving for m gives

$$\tfrac{11}{2}m = 5 \Rightarrow m = \tfrac{10}{11}$$

Multiplying by 60 to convert to seconds gives a value of $\frac{600}{11} = 54\frac{6}{11}$. Since $\frac{6}{11} > \frac{1}{2}$, we round up to 55 seconds.

Answer is (E).

Senior Final, Part B

1. (a) Expanding the expression gives

$$(r+1)(r+2)(r-4) = r^3 - r^2 - 10r - 8 = r\left(r^2 - r - 10\right) - 8 = -8$$

Alternative solution

We have

$$r^2 - r - 10 = 0 \Leftrightarrow r^2 = r + 10$$

so

$$\begin{aligned}(r+1)(r+2)(r-4) &= \left(r^2 + 3r + 2\right)(r-4) = (r + 10 + 3r + 2) = 4(r+3)(r-4)\\ &= 4\left(r^2 - r - 12\right) = 4\left(r^2 - r - 10 - 2\right) = -8\end{aligned}$$

Answer: The value is -8.

(b) If n is an odd positive integer 14^n has a 4 in the ones place. Hence $11 \cdot 14^n$ also has a 4 in the ones place and so $11 \cdot 14^n + 1$ has a 5 in the ones place. Every such number is divisible by 5 and is greater than 5, so it is not prime. Therefore, there are no numbers of the form $11 \cdot 14^n + 1$ that are prime numbers.

Answer: None

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2. See the solution to Junior Final, Part B, Problem 4.

Answer: The numbers are $-3, 3, 4, 5, 7$.

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3. Plugging in $x = \frac{3}{2}$ gives

$$\begin{aligned}F\left(\tfrac{3}{2}\right) &= \frac{1}{\sqrt{\frac{3}{2} + 2\sqrt{\frac{1}{2}}}} + \frac{1}{\sqrt{\frac{3}{2} - 2\sqrt{\frac{1}{2}}}}\\ &= \frac{1}{\sqrt{\frac{3}{2} + \sqrt{2}}} + \frac{1}{\sqrt{\frac{3}{2} - \sqrt{2}}}\\ &= \frac{\sqrt{2}}{\sqrt{3 + 2\sqrt{2}}} + \frac{\sqrt{2}}{\sqrt{3 - 2\sqrt{2}}}\\ &= \sqrt{2}\left(\frac{\sqrt{3 - 2\sqrt{2}} + \sqrt{3 + 2\sqrt{2}}}{\sqrt{9 - 8}}\right)\\ &= \sqrt{2}\left(\sqrt{3 - 2\sqrt{2}} + \sqrt{3 + 2\sqrt{2}}\right)\end{aligned}$$

Hence,

$$\left[F\left(\tfrac{3}{2}\right)\right]^2 = 2\left(3 - 2\sqrt{2} + 2\sqrt{9 - 8} + 3 + 2\sqrt{2}\right) = 16$$

Since $F\left(\frac{3}{2}\right) > 0$, taking the positive square root gives $F\left(\frac{3}{2}\right) = 4$.

Answer: The integer is 4.

4. There are two steps:

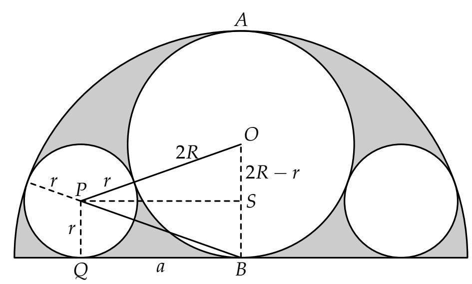
   - Determine the number of ways in which the student receiving the correct letter can be chosen.
   - Determine the number of ways in which the four remaining letters can be sent to the wrong student.

   The first step can be done in 5 ways. The second step is the same as the problem in Junior Final, Part A, Problem 7, which shows that there are 9 ways to send the four remaining letters to the wrong student. Thus, the total number of ways in which all but one student receives the letter for a different student is $5 \times 9 = 45$.

   Answer: There are 45 ways.

---

5. Let O and P be the centres of the large and small unshaded circles, respectively, as shown. The line through P perpendicular to the diameter of the semicircle intersects the diameter at Q. The line PS is parallel to the diameter of the semicircle. Let r be the radius of the small unshaded circle and $2R$ the radius of the large unshaded circle. Then the radius of the semicircle is $4R$. Finally, let a be the length of the segment QB, as shown. Now the radius of the semicircle through point P intersects the semicircle at the point of tangency of the small circle and semicircle, so that $\overline{BP} = 4R - r$. Similarly, $\overline{OP} = 2R + r$. Then applying Pythagoras' Theorem to triangles BQP and OPS gives:



$$(4R - r)^2 = r^2 + a^2 \quad \text{and} \quad (2R + r)^2 = (2R - r)^2 + a^2$$

The difference of these two equations gives

$$\begin{aligned} &(4R - r)^2 - (2R + r)^2 = r^2 - (2R - r)^2 \\ \Rightarrow\ &16R^2 - 8Rr + r^2 - 4R^2 - 4Rr - r^2 = r^2 - 4R^2 + 4Rr - r^2 \\ \Rightarrow\ &16R^2 - 16Rr = 16R\,(R - r) = 0 \Rightarrow r = R \end{aligned}$$

since $R \neq 0$. Hence, if A is the area of one of the small circles, then the area of the semicircle is $8A$ and the area of the large circle is $4A$. So the area of the shaded region is $8A - 4A - 2A = 2A$. So $\frac{1}{4}$ of the semicircle is shaded.

Answer: $\frac{1}{4}$ is shaded.