BRITISH COLUMBIA SECONDARY SCHOOL MATHEMATICS CONTEST, 2008

Solutions

Junior Preliminary

1. We compute, $\dfrac{1}{10} + \dfrac{9}{100} + \dfrac{7}{10000} = 0.1 + 0.09 + 0.0007 = 0.1907$.

Answer is (C).

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2. Simple calculation gives

$$\frac{1}{3 + \dfrac{1}{2 + \dfrac{1}{2}}} = \frac{1}{3 + \dfrac{2}{5}} = \frac{5}{17}$$

Answer is (B).

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3. If X, Y and Z denote the areas of the outer, middle, and inner squares, respectively, then the required ratio is $\dfrac{Y - Z}{X} = \dfrac{3^2 - 1^2}{4^2} = \dfrac{8}{16} = \dfrac{1}{2}$.

Answer is (C).

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4. Making the substitution gives

$$\boxed{y+3} = 5\,(y+3)^2 - (y+3) = 5\left(y^2 + 6y + 9\right) - y - 3 = 5y^2 + 29y + 42$$

Alternate solution: If $y = 0$, then $\boxed{y+3} = \boxed{3} = 5 \times 3^2 - 3 = 42$. The only one of the choices with a value of 42 when $y = 0$ is $5y^2 + 29y + 42$.

Answer is (A).

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5. We compute, $10^{-49} - [2 \text{ x } 10^{-50}] = 10^{-50}[10 - 2] = 8 \text{ x } 10^{-50}$.

Answer is (E).

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6. If x is the two digit number, then we have the sequence of transformations

$$x \longrightarrow 10x + 6 \longrightarrow 10x + 6 + 6 \longrightarrow 10(x+1) + 2 \longrightarrow x + 1 = 76$$

Thus, $x = 75$.

Answer is (D).

7. Since the distance d traveled in time t at speed s is given by $d = st$, the time to travel distance d is $t = \frac{d}{s}$. So the total time the bug takes for his run is

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} = \frac{49}{20} \text{ seconds}$$

The total distance is 6 centimetres, so

$$\text{average speed} = \frac{6}{\frac{49}{20}} = \frac{120}{49} \text{ centimetres per second}$$

Answer is (D).

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8. The diameter of the smaller circle is the side of the square $= s$ while the diameter of the larger circle is the diagonal of the square $= \sqrt{2}s$. The ratio of the areas of the circles is the ratio of the squares of the diameters, i.e. 1 : 2.

Answer is (A).

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9. The given equation has exactly one real root if and only if its discriminant $D = B^2 - 8$ is zero. Clearly $B = \pm\sqrt{8}$ in this case.

Answer is (E).

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10. If n is the original number of cows, p is the price paid for each cow, and q is the price for which each cow was sold, then we are given $np = 480$, $(n-3)q = 495$, and $q = p + 1 \Rightarrow p = q - 1$. Then, $nq - n = 480$ and $nq - 3q = 495$. Eliminating nq gives $n = 3q + 15$. Substituting into the equation involving n and q gives $q\,(3q + 12) = 495$, or $q^2 + 4q - 165 = (q + 15)\,(q - 11) = 0$. Since q must be positive we conclude that $q = 11$, so that $n = 48$.

Answer is (D).

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11. If H and h denote, respectively, the heights of the similar triangles ABC and ADE, then

$$\frac{H}{h} = \frac{\overline{BC}}{\overline{DE}}$$

Since the area of the triangle ABC is twice the area of triangle ADE, we have

$$\frac{\frac{1}{2}H \times \overline{BC}}{\frac{1}{2}h \times \overline{DE}} = \frac{H \times \overline{BC}}{h \times \overline{DE}} = 2 \Rightarrow \frac{H}{h} = 2\frac{\overline{DE}}{\overline{BC}}$$

Hence we have

$$\frac{\overline{BC}}{\overline{DE}} = 2\frac{\overline{DE}}{\overline{BC}} \Rightarrow \overline{DE}^2 = \frac{1}{2}\overline{BC}^2 = \frac{1}{2}\left(2^2\right) = 2 \Rightarrow \overline{DE} = \sqrt{2}$$

Answer is (B).

12. The pattern of $\times$'s repeats every four rows and for every block of four rows there are three numbers in each column. Further

$$\frac{50}{3} = 16\tfrac{2}{3}$$

Thus, the fiftieth number in the second, or any, column is the second number in that column in the seventeenth block. Since there are 12 numbers in each block, the difference between the numbers in corresponding positions in two successive blocks is 12. Thus, the second number in the second column of the n^{th} group is $5 + 12(n-1)$. For $n = 17$ this gives $5 + 12 \times 16 = 197$. This is the fiftieth number in the second column.

Answer is (D).

Senior Preliminary

1. Let V be the total number of votes. Then $0.6V - 0.4V = 55$ giving $V = 275$.

Answer is (C).

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2. Completing the square in the quadratic gives

$$5x^2 - 4x + c = 5\left(x^2 - \tfrac{4}{5}x + \tfrac{4}{25} - \tfrac{4}{25}\right) + c = 5\left(x - \tfrac{2}{5}\right)^2 - \tfrac{4}{5} + c$$

For the graph to touch the x-axis exactly once, we must have $-\frac{4}{5} + c = 0 \Rightarrow c = \frac{4}{5}$.
Alternate solution I: Solutions to the quadratic equation $5x^2 - 4x + c = 0$ are given by

$$x = \frac{4 \pm \sqrt{16 - 20c}}{10}$$

The graph of the equation $y = 5x^2 - 4x + c$ touches the x-axis exactly once if the quadratic equation has only one solution. This is the case if the discriminant $16 - 20c = 0$. Solving for c gives $c = \frac{4}{5}$.

Alternate solution II: The graph of the quadratic function $y = ax^2 + bx + c$ has vertex at $\left(-\dfrac{b}{2a}, \dfrac{b^2 - 4ac}{4a}\right)$. The graph touches the x-axis exactly once if the y-coordinate of the vertex is zero. Here this gives $16 - 20c = 0 \Rightarrow c = \frac{4}{5}$.

Answer is (D).

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3. Let d be the number of dimes. Then the value of dimes, in cents, is $10d$. The number of nickels is $3d$ and their value is $5 \times 3d = 15d$. The value of the pennies is $6 + d$. Hence, the total value of the coins is

$$10d + 15d + 6 + d = 838 \Rightarrow 26d = 832 \Rightarrow d = 32$$

Alternate Solution: Get rid of the 6 extra pennies to leave 832 cents. Then there would be batches of 3 nickels, 1 dime, 1 penny in each batch, totaling 26 cents. Since there is 1 dime per batch, the number of dimes is $\dfrac{832}{26} = 32$.

Answer is (A).

4. Consider the radius of the large circle that goes through the centre of one of the small circles, as shown, where x is the length of the segment radius that does not lie in the small circle. If r is the radius of each small circle, then $2r + x = 1$. Further, by symmetry the angle, θ, between the radius through the centre of the small circle and the radius that is tangent to the two circles is $\theta = 45°$. Hence, $r + x = \sqrt{2}r$. Thus,

$$r + \sqrt{2}r = 1 \Rightarrow r = \frac{1}{\sqrt{2}+1} = \sqrt{2} - 1$$

Answer is (B).

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5. We have $\frac{d}{t} = c$ for some positive constant c, so $d = ct$ for $0 < t \leq 5$. This is the slope-intercept equation of a line of slope c through the origin. Also, on $5 \leq t \leq 10$, d remains constant. These considerations tell us the correct graph is choice (D).

Answer is (D).

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6. If the woman starts at the origin $(0, 0)$ then after the first leg of her trip she is at $(1, 0)$, after the second she is at $(1 + \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ and after the third she is at $(2 + \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$. The distance which she went is

$$\sqrt{\left(2 + \frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2} = \sqrt{5 + 2\sqrt{2}}$$

Answer is (B).

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7. Let n be Antonino's number. The number of numbers less than or equal to n that are divisible by 9 is no larger than $n/9$, the number divisible by 11 is no larger than $n/11$, and the number divisible by both, that is, divisible by 99, is no larger than $n/99$. By the inclusion-exclusion principle the number of numbers divisible by both 9 and 11 is the number divisible by 9 plus the number divisible by 11 minus the number divisible by both, to eliminate double counting. So the number of numbers less than or equal to n that are divisible by 9 and 11, but not both is no larger than

$$\frac{n}{9} + \frac{n}{11} - \frac{2n}{99} = \frac{18n}{99} = \frac{2n}{11}$$

Hence, Antonino's number satisfies the inequality

$$\frac{2n}{11} \leq 100 \Rightarrow n \leq 550$$

Now $550/9 = 61+$, $550/11 = 50$, and $550/99 = 5+$ so that number of numbers less than or equal to 550 that are divisible by 9 or 11, but not both is $61 + 50 - 2 \times 5 = 101$. However, this includes 550. So there are exactly 100 numbers strictly less than 550 that are divisible by 9 or 11, but not both. Further, there are 101 numbers less than 551 divisible by 9 or 11, but not both. Hence, Antonino's number is 550.

Answer is (D).

8. The number of ways to divide the CDs can be determined by thinking of labeling each CD with either the letter T for Tom or J for Jerry. For each CD there are two choices, so by the multiplication principle the total number of ways of dividing the CDs is $2^6 = 64$. But, this includes the two cases in which all of the CDs are labeled with a T or with a J, so that one person gets no CDs. So the number of ways to label the CDs so that each person gets at least one CD is $64 - 2 = 62$.

Answer is (E).

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9. Note that $\sin(90° + x) = \cos x$. So that, for example $\sin(100°) = \cos(10°)$. Hence,

$$\begin{aligned}&\sin^2(10°) + \sin^2(20°) + \cdots + \sin^2(170°)\\ &\quad = \left[\sin^2(10°) + \cos^2(10°)\right] + \cdots + \left[\sin^2(80°) + \cos^2(80°)\right] + \sin(90°)\\ &\quad = 9\end{aligned}$$

since $\cos^2 x + \sin^2 x = 1$ for all x, and $\sin(90°) = 1$.

Answer is (D).

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10. If Garfield starts with lasagna number one and goes around the circle, eating lasagnas as described he eats them in the order: 1, 2, 4, 7, 3, 5, 6. In particular, the last lasagna he will eat is number 6. If he starts at 2, just increase each of the numbers in the list above up one (if you go above 7, just subtract 7). Thus, the fourth lasagna he eats is number 1 and he eats number 7 last. If he starts at 3, the seventh lasagna he eats is number 1. Thus, if Garfield starts at lasagna 3 he will eat number 1 last.

Answer is (B).

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11. Let O be a vertex of the face containing the point Q. Drop a perpendicular line from point Q to one of the edges containing the point O and let P be the point where this line intersects the edge. Consider the right triangle POR in which sides OR and OP are perpendicular with $\overline{OP} = 2$ and $\overline{OR} = 4$. The length of the hypotenuse PR is

$$\overline{PR} = \sqrt{\overline{OP}^2 + \overline{OR}^2} = \sqrt{4+16} = 2\sqrt{5}$$

Now consider the right triangle QPR with sides QP and PR perpendicular with $\overline{QP} = 2$. The length of the hypotenuse QR, the distance we want, is

$$\overline{QR} = \sqrt{\overline{QP}^2 + \overline{PR}^2} = \sqrt{4+20} = 2\sqrt{6}$$

Alternate solution: Set up a three dimensional coordinate system with the x-axis horizontal, the y-axis vertical, and the z-axis pointing toward you. If one vertex of the cube is at the origin and Q is in the xy-plane, Q has coordinates $(2,2,0)$. Then, the point R can have coordinates $(0,0,4)$. The distance between Q and R is given by the three dimensional form of Pythagoras' theorem as

$$\sqrt{(2-0)^2 + (2-0)^2 + (0-4)^2} = \sqrt{4+4+16} = \sqrt{24} = 2\sqrt{6}$$

Answer is (B).

12. Since $2008 = 2^3 \times 251$ where 251 is prime. The possible sets of values for a, b, c, and d are

$$\{1,1,1,2008\}, \{1,1,2,1004\}, \{1,1,4,502\}, \{1,1,8,251\},$$
$$\{1,2,2,502\}, \{1,2,4,251\}, \{2,2,2,251\}$$

Each of these sets gives a different value for the sum $a+b+c+d$. So there are 7 possible values for the sum.

Answer is (D).

Junior Final, Part A

1. Let x be the value of the car. Then

$$\frac{7}{12}(x+8000) = x+1600 \Rightarrow 7x+56000 = 12x+19200 \Rightarrow 5x = 36800 \Rightarrow x = 7360$$

Answer is (C).

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2. We can omit any number that is not a multiple of three, leaving:

$$3 \times 6 \times 9 \times 12 \times 15 \times 18 \times 21 \times 24 \times 27 \times 30$$

But then, we can also omit factors which are not 3, leaving:

$$3 \times 3 \times 9 \times 3 \times 3 \times 9 \times 3 \times 3 \times 27 \times 3 = 3^{14}$$

Thus, $x = 14$.
Alternate solution: There are 10 multiples of 3 between 1 and 30, 3 multiples of $9 = 3^2$, and one multiple of $27 = 3^3$. Each multiple of 9 is already counted once in the multiples of 3, and 27 is counted as a multiple of 3 and of 9. This gives a total of $x = 10+3+1 = 14$ factors of 3 in multiplying out 30!.

Answer is (B).

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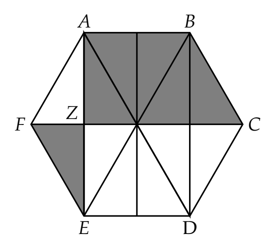
3. If we draw additional line segments, as shown, we obtain 12 congruent triangles. From the diagram we see that the ratio is 1 : 5.



Answer is (A).

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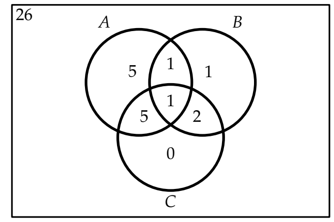
4. Let $z = n + d$, where n is an integer and d is a real number with $0 < d < 1$, which is the decimal part of the number z. Then $\lfloor z \rfloor = n$ and $\lfloor 1-z \rfloor = \lfloor -n+1-d \rfloor = -n$. Hence, $\lfloor z \rfloor + \lfloor 1-z \rfloor = 0$.

Answer is (B).

5. Consider the Venn diagram shown. Working from the center out, with ABC being 1, we can fill in the values as shown. Then we see that a total of 15 students failed at least one subject, so there were $41 - 15 = 26$ students who passed all three subjects.



Answer is (E).

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6. In counting the triangles we can put the vertices in alphabetical order.

First side	Other vertex	Number
AB, AC, BC	D, E, F	9
AD, BD, CD	E, F	6
AE, BE, CE, DE	F	4
		19

Alternate solution: To form a triangle we need three points. The number of ways to choose three points from the six points is

$$\binom{6}{3} = \frac{6 \times 5 \times 4}{3 \times 2 \times 1} = 20$$

One of these choices, ABC, does not form a triangle since A, B, and C are on a straight line. So 19 triangles can be formed.

Answer is (D).

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7. Let $R = \overline{BC} = \overline{CD}$ be the radius of the circle. Since $\angle ADC$ is a right angle we can we Pythagoras' theorem on triangle ACD:

$$(R+7)^2 = R^2 + 10^2 \Rightarrow R^2 + 14R + 49 = R^2 + 100 \Rightarrow 14R = 51 \Rightarrow R = \frac{51}{14}$$

Answer is (C).

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8. We are only interested in the last digit, and we note that the powers of 8 end in the digits 8, 4, 2, and 6 and that this pattern repeats with period 4. Since $2008 = 4 \times 502$, 2008 is the last number in a group of 4, so the last digit must be a 6.

Answer is (B).

9. From the table below we see that the values of M which have the sum and difference both primes are 16, 25, and 29. The sum of these numbers is $16 + 25 + 29 = 70$.

M	11	13	16	17	19	23	25	27	29	31	32	37
Sum	2	4	7	8	10	5	7	9	11	4	5	10
Difference	0	2	5	6	8	1	3	5	7	2	1	4

Alternate solution: A number that is divisible by only one prime must either be prime or some power of a single prime. Further, any prime number, except 2, is odd, so we can reject any prime for which the first digit is odd, except 11, otherwise the sum of the digits is even and so not prime. This means that the only numbers we need to consider are 11, 16, 23, 25, 27, 29, and 32. For three of these, 11, 23 and 32, the difference between the digits is 0 or 1, which are not prime. For 27 the sum of the digits is 9, which is not prime. The three remaining numbers, 16, 25, and 29 satisfy all of the requirements.

Answer is (D).

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10. Let h be the length of the segments DE and BF, and let x be the length of the side AB. Then, using Pythagoras, the length of the diagonal AC is $\sqrt{1+x^2}$. Since $\overline{AE} = \overline{EF} = \overline{FC}$, we have

$$\overline{AE} = \frac{1}{3}\overline{AC} = \frac{1}{3}\sqrt{1+x^2}$$

and

$$\overline{AF} = \frac{2}{3}\overline{AC} = \frac{2}{3}\sqrt{1+x^2}$$

Then applying Pythagoras to triangle ADE gives

$$h^2 + \left(\frac{1}{3}\sqrt{1+x^2}\right) = 1 \Rightarrow h^2 + \frac{1+x^2}{9} = 1 \Rightarrow x^2 + 9h^2 = 8$$

and applying Pythagoras to triangle AFB gives

$$h^2 + \left(\frac{2}{3}\sqrt{1+x^2}\right) = x^2 \Rightarrow h^2 + \frac{4\left(1+x^2\right)}{9} = x^2 \Rightarrow 5x^2 - 9h^2 = 4$$

Adding the two equations above, cancelling the $9h^2$, gives $6x^2 = 12 \Rightarrow x^2 = 2 \Rightarrow x = \sqrt{2}$.

Answer is (A).

Junior Final, Part B

1. We are looking for the coordinates of the point on the line $4x + 3y = 12$ closest to the origin. We know that the closest such point occurs where the line through the origin $y = mx$ and $y = -\frac{4}{3}x + 4$ are perpendicular. To be perpendicular, the slopes m and $-\frac{4}{3}$ must be negative reciprocals:

$$m \times -\frac{4}{3} = -1$$

which implies that

$$m = \frac{3}{4}$$

Hence, the line $y = \frac{3}{4}x$ is the required line through the origin. These lines intersect when

$$\frac{3}{4}x = -\frac{4}{3}x + 4$$

Solving for x, we obtain $x = \frac{48}{25}$. Substituting the resulting x value into $y = \frac{3}{4}x$, we obtain $y = \frac{3}{4} \times \frac{48}{25} = \frac{36}{25}$. Hence, the coordinates are $\left(\frac{48}{25}, \frac{36}{25}\right)$.

Answer: The coordinates of the point are $\left(\frac{48}{25}, \frac{36}{25}\right)$.

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2. Let n be the amount by which the length of the base is decreased. We wish to find the value of n. Since we want the new area to be half of the original, i.e, $A_n = \frac{1}{2}A_0 = \frac{1}{4}bh$, we have

$$\begin{aligned}\frac{1}{4}bh = \frac{1}{2}(b-n)(h+m) = \frac{1}{2}(bh + bm - nh - nm) &\Rightarrow bh = 2(bh + bm - nh - nm)\\ \Rightarrow bh = 2bh + 2bm - 2nh - 2nm &\Rightarrow 0 = bh + 2bm - 2n(h+m)\\ \Rightarrow bh + 2bm = 2n(h+m) &\Rightarrow n = \frac{b(h+2m)}{2(h+m)}\end{aligned}$$

For the new area to be half of the original, the base must be decreased by $n = \dfrac{b(h+2m)}{2(h+m)}$.

Answer: The base must be decreased by $n = \dfrac{b(h+2m)}{2(h+m)}$

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3. Consider the sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ where

$$a_1 = 2,\ a_2 = 2 + \frac{1}{a_1},\ a_3 = 2 + \frac{1}{a_2}, \ldots, a_n = 2 + \frac{1}{a_{n-1}}$$

(a) Using the relations given

$$a_2 = 2 + \frac{1}{2} = \frac{5}{2},\ a_3 = 2 + \frac{1}{5/2} = \frac{12}{5},\ a_4 = 2 + \frac{1}{12/5} = \frac{29}{12}$$

Answer: The respective values are $\frac{5}{2}$, $\frac{12}{5}$, and $\frac{29}{12}$.

Problem 3 continued ...

...Problem 3 continued

(b) Since a is the value that a_n approaches, replacing both a_{n-1} and a_n by a in the recurrence relation above gives

$$a = 2 + \frac{1}{a} \Rightarrow a^2 - 2a - 1 = 0$$

Using the quadratic formula to solve this equation for a gives

$$a = \frac{2 \pm \sqrt{4 - 4(1)(-1)}}{2} = \frac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2}$$

Since a must be positive, we see that $a = 1 + \sqrt{2}$.

Answer: The values of a_n approach $1 + \sqrt{2}$.

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4. Let $x = \cdots cba$, where a is the ones digit of x, b is the tens digits, and so on. Then $x = \cdots + 100c + 10b + a$. Expanding x^2 gives

$$x^2 = \cdots + 10(2ab) + a^2$$

From this we see that the ones digit of x^2 is the ones digit of a^2 (the square of the ones digit of x), and that the tens digit of x^2 is even, unless there is an odd carry from a^2. Considering the squares of the integers from 1 to 9, we see that there is an odd carry only for 4 and 6. For both of these the ones digit of the square is 6.

Answer: 6 is the only possible value of the ones digit.

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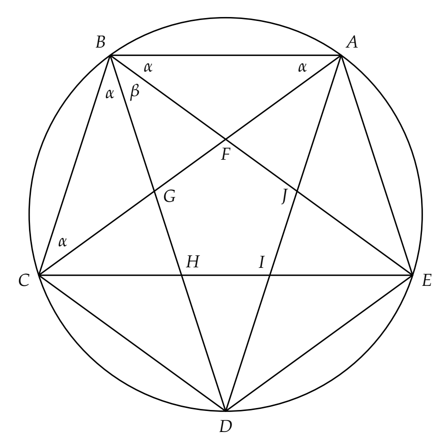
5. Five points $ABCDE$ are equally spaced around a circle and a line segment is drawn from each point to the other four points.

(a) The diagram is shown. Since this is a regular pentagon, the interior angle, θ, at each of the vertices A, B, C, D, and E will be given by $5\theta = 5 \times 180° - 360° = 540° \Rightarrow \theta = 108°$. It is apparent from the diagram that one of the angles α or β is the smallest angle formed. But, from the previous observation



$$2\alpha + \beta = 108°$$

Further, considering the triangle ABC we see that

$$4\alpha + \beta = 180°$$

Subtracting these equations gives $2\alpha = 72° \Rightarrow \alpha = 36°$. This gives $\beta = 36°$. Thus, the smallest angle formed is $36°$.

Answer: The smallest angle formed is $36°$.

(b) There are 5 large triangles like ABC and 5 large triangles like ACD for a total of 10 triangles that contain only three of the outer vertices A, B, C, D, and E. Then there are 5 small triangles like ABF, 10 triangles like ABJ and AEF, and 5 triangles like ACI for a total of 20 triangles that contain two of the outer vertices. Finally, there are 5 small triangles like AFJ that contain only one of the outer vertices. There are no triangles that contain none of the outer vertices. This gives a total 35 triangles.

Answer: There are 35 triangles formed.

Senior Final, Part A

1. Using the definitions given we have

$$\frac{(2n+1)!}{(2n+1)\text{¡}} = \frac{1 \times 2 \times 3 \times 4 \times \cdots \times 2n \times (2n+1)}{1 \times 3 \times 5 \times 7 \times \cdots \times (2n+1)} = 2 \times 4 \times 6 \times \cdots \times 2n = 2^n n!$$

Answer is (A).

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2. If P is anywhere in the square, then both $\angle PDC$ and $\angle PCD$ have measures less than or equal to $90°$, that is, they are acute angles. Draw the semicircle with diameter DC as shown. If P lies on the semicircle, then $\angle CPD$ is a right angle. If P is outside the semicircle, then $\angle CPD$ is acute. If P is inside the semicircle, $\angle CPD$ is greater than $90°$, that is, it is an obtuse angle. The probability that $\angle CPD$ is an acute angle is equal to the area of the part of the square which is above the circle or

$$1 - \frac{\pi}{8} = \frac{8 - \pi}{8}$$

Answer is (D).

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3. The length of AB is $\sqrt{6^2 + 3^2 + 3^2} = \sqrt{54} = 3\sqrt{6}$. By similar triangles half of its length lies in the $3 \times 3 \times 3$ cube, so the length of the portion of the line segment AB that lies in the $3 \times 3 \times 3$ cube is $\frac{3\sqrt{6}}{2}$.

Answer is (C).

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4. Write the equation as

$$x^3 - x^2 = y^2 \Rightarrow x^2(x-1) = y^2$$

In order for the left-hand side to be a perfect square, we must have $x - 1 = k^2$ for some integer k. Since any integer k will work, there is an infinite number of possibilities for the value of x and for the ordered pair (x, y).

Answer is (D).

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5. The solutions to this inequality are the points that lie inside the square in the xy-plane with vertices at $(50, 0)$, $(0, 50)$, $(-50, 0)$, and $(0, -50)$. Along the line $x = 0$ we have $y = 1, 2, \ldots, 49$, and along the line $x = 1$ we have $y = 1, 2, \ldots, 48$, and so on for a total of $49 + 48 + \cdots + 1 = \dfrac{49 \times 50}{2}$ points in the first quadrant including the positive y-axis. Multiply by four to get 4900. But this misses the origin, so the total is 4901.

Answer is (B).

6. Each square has sides $\sqrt{2}/2$ times the previous square. The sum of the perimeters is a geometric series with first term $a = 4$ and common ratio $r = \sqrt{2}/2$ which has sum

$$\frac{a}{1-r} = \frac{4}{1-\sqrt{2}/2} = \frac{4(1+\sqrt{2}/2)}{1/2} = 8 + 4\sqrt{2}$$

Answer is (C).

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7. Regroup to get

$$\begin{aligned}\left(21^2 + 22^2 + 23^2 + \cdots + 40^2\right) - \left(20^2 + 19^2 + 18^2 + \cdots + 1^1\right)&\\ &= 21^2 - 1^2 + 22^2 - 2^2 + 23^2 - 3^2 + \cdots + 40^2 - 20^2\\ &= 20 \times 22 + 20 \times 24 + 20 \times 26 + \cdots + 20 \times 60\\ &= 20\,(22 + 24 + 26 + \cdots + 60)\\ &= 20 \times 20 \times \frac{22+60}{2}\\ &= 16400\end{aligned}$$

Answer is (B).

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8. We know by the binomial theorem that

$$(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$

The constant term will be

$$15a^4b^2 = 15\,(3x)^4\left(\frac{2}{x^2}\right)^2 = 15 \times 3^4 \times 2^2 = 4860$$

Answer is (D).

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9. We know that $2^0 + 2^1 + 2^2 + 2^3 = 1 + 2 + 4 + 8 = 15$. Since $2008 \equiv 0 \pmod 4$ we have

$$2^0 + 2^1 + 2^2 + \cdots + 2^{2008} \equiv 1 \pmod{15}$$

Answer is (B).

10. Assume that the radius of the circle is 1. The triangles COP and DOQ are equilateral triangles with side length 1. The area of the unshaded region between the line DQ and the arc DQ is the difference between the areas of the circular segment DOQ and the triangle DOQ, which is

$$A_{DQ} = \frac{\pi}{6} - \frac{\sqrt{3}}{4}$$

The area of the unshaded region between the line PC and the arc PC, A_{PC}, is the same. The area of the shaded region A_2 is the area of the semicircle reduced by the sum of the two areas A_{DQ} and A_{PC}. This gives

$$A_2 = \frac{\pi}{2} - 2\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right) = \frac{\pi}{6} + \frac{\sqrt{3}}{2}$$

The area of the unshaded region A_1 is the difference between the area of the triangle PQR and A_2. This gives

$$A_1 = \sqrt{3} - \left(\frac{\pi}{6} + \frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{2} - \frac{\pi}{6}$$

Thus the ratio of the area A_1 to area A_2 is

$$\frac{A_1}{A_2} = \frac{\dfrac{\sqrt{3}}{2} - \dfrac{\pi}{6}}{\dfrac{\sqrt{3}}{2} + \dfrac{\pi}{6}} = \frac{3\sqrt{3} - \pi}{3\sqrt{3} + \pi}$$

Answer is (E).

Senior Final, Part B

1. (a) The simplified form of the expression is

$$\frac{1}{a-1} + \frac{1}{a+1} = \frac{a+1+a-1}{(a-1)(a+1)} = \frac{2a}{(a-1)(a+1)} = \frac{2a}{a^2-1}$$

Answer: The simplified form is $\frac{2a}{a^2-1}$.

(b) Since $a^2 - 1 < a^2$, for any a, we have, for $a > 1$,

$$\frac{1}{a^2-1} > \frac{1}{a^2} \Rightarrow \frac{1}{a-1} + \frac{1}{a+1} = \frac{2a}{a^2-1} > \frac{2a}{a^2} = \frac{2}{a}$$

Therefore, $\frac{1}{a-1} + \frac{1}{a+1}$ is larger than $\frac{2}{a}$ for $a > 1$.

Answer: $\frac{1}{a-1} + \frac{1}{a+1}$ is larger than $\frac{2}{a}$ for $a > 1$.

(c) For $a = 1000000$ in part (b) we have $\frac{2}{a} = \frac{1}{500000}$. Using the result in part (b) we get

$$\frac{1}{1000000-1} + \frac{1}{1000000+1} > \frac{1}{500,000}$$

Answer: The value of $\frac{1}{999999} + \frac{1}{1000001}$ is larger than $\frac{1}{500,000}$.

2. (a) Given a circle of radius a with centre at $(0, a)$ and a parabola $y = x^2$. Where do these two conics meet, knowing that the circle has a standard form of $x^2 + (y - a)^2 = a^2$? Solving for x^2 in the equation of the circle, we obtain

$$x^2 = -y^2 + 2ay$$

Setting $x^2 = y$ and $x^2 = -y^2 + 2ay$ equal to one another, we obtain

$$\begin{aligned} y^2 + y - 2ay &= 0 \\ y^2 + y(1 - 2a) &= 0 \\ y[y + (1 - 2a)] &= 0 \end{aligned}$$

Hence, either $y = 0$ or $y = -(1 - 2a) = 2a - 1$ or both. When $y = 0$, $x^2 = y$ implies that $x^2 = 0$ and $x = 0$. When $y = 2a - 1$, then $x^2 = 2a - 1$ and hence $x = \pm\sqrt{2a - 1}$.

Answer: The points of intersection are $(0, 0)$, $\left(\sqrt{2a-1}, 2a - 1\right)$, **or** $\left(-\sqrt{2a-1}, 2a - 1\right)$.

(b) If the circle is centred on the y-axis at $(0, a)$ and it goes through the origin, then its radius must be $r = a$. Hence, we can use the results in part (a) above. In particular, note that if $a \le \frac{1}{2}$, the only point of intersection between the circle and the parabola is the origin. For any larger value of a, there are three points of intersection. Thus the largest radius is $r = a = \frac{1}{2}$.

Answer: The largest radius is $\frac{1}{2}$.

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3. Let y, m, and o represent the ages of the youngest, middle one, and the oldest residents on the house. Let N be the age of the neighbour and H the house number. So, we have

- $y \le m \le o \le N$
- $y \times m \times o = 252 = 2^2 \times 3^2 \times 7$
- $y + m + o = H$

Since $y \times m \times o = 252$, we have $1 \le y \le m \le o \le N$. We will construct all possible combinations of y, m, o in the table below. First note that the factors of 252 are $1, 2, 3, 4, 6, 7, 9, 12, 14, 18, 21, 28, 36, 42, 63, 84, 126, 252$.

y	m	o	H	y	m	o	H
1	1	252	254	2	2	63	67
1	2	126	129	2	3	42	47
1	3	84	88	2	6	21	29
1	4	63	68	2	7	18	27
1	6	42	49	2	9	14	25
1	7	36	44	3	3	28	34
1	9	28	38	3	4	21	28
1	12	21	34	3	6	14	23
1	14	18	33	3	7	12	22
				4	7	9	20
				6	6	7	19

Notice that if the house number appears in the table only once, then the census taker did not have to ask the last question (as he was very intelligent). So the house number H must appear in the table at least twice.

Problem 3 continued ...

...Problem 3 continued

By examining the table, we see that the house number is 34 appears twice, and that this is the only house number appearing at least twice. So, the ages are either

case A: $1, 12, 21$, or case B: $3, 3, 28$.

Now, if the age N of the neighbour were 32, for example, then the answer to the last census taker's question does not help the census taker to determine the case A or B. So, the fact that "nobody is older than the neighbour" gives the exact ages of the residents tells us that the neighbour's age N must be $21 \leq N < 28$. It follows that the ages of residents must be 1, 12, and 21.

Answer: The ages of residents must be 1, 12, and 21.

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4. In order for the given numbers to be the lengths of the three sides of a triangle, we must have

$$k < \frac{1}{k} + 2, \text{ and } \frac{1}{k} < k + 2, \text{ and } 2 < k + \frac{1}{k}$$

Note that k must be positive, so we can simplify each of these inequalities by multiplying through by k. The first inequality gives

$$k^2 < 1 + 2k \Rightarrow k^2 - 2k - 1 < 0$$

The quadratic expression $k^2 - 2k - 1$ is negative between the two roots of the quadratic equation $k^2 - 2k - 1 = 0$, which are $k = 1 \pm \sqrt{2}$. Only $k = 1 + \sqrt{2}$ is positive, so the first inequality gives $0 < k < 1 + \sqrt{2}$. The second inequality gives

$$1 < k^2 + 2k \Rightarrow k^2 + 2k - 1 > 0$$

The quadratic expression $k^2 + 2k - 1$ is positive to the right of or to the left of the two roots of the quadratic equation $k^2 + 2k - 1 = 0$, which are $k = -1 \pm \sqrt{2}$. Only $k = -1 + \sqrt{2}$ is positive, so the first inequality gives $1 - \sqrt{2} < k$. The third inequality gives

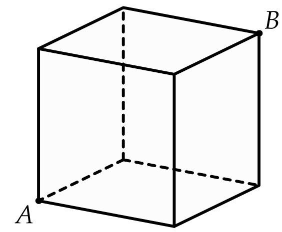
$$2k < 1 + k^2 \Rightarrow k^2 - 2k + 1 = (k-1)^2 > 0$$

Since $(k-1)^2$ is positive for all k except $k = 1$, the third inequality gives $0 < k < 1$ or $k > 1$. Combining the three sets of inequalities we obtain

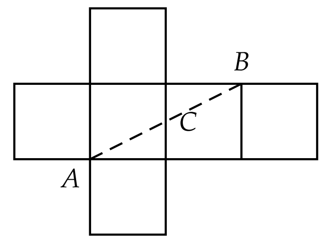
$$\sqrt{2} - 1 < k < 1 \text{ or } 1 < k < \sqrt{2} + 1$$

Answer: The number k must satisfy the inequalities $\sqrt{2} - 1 < k < 1$ or $1 < k < \sqrt{2} + 1$.

5. The shortest path on the surface of a cube from vertex A to the furthest vertex B involves crossing a certain number of faces and edges of the cube. See the diagram.



(a) If we open the cube, the given cube would look this. The shortest path from the point A to the point B is given by the straight line connecting the two points. Thus, we have to cross two faces and one edge.



Answer: The shortest path crosses two faces and one edge.

(b) Consider the cube shown above. The path shown in part (a) above goes through the point C. Another path with the same length goes through the point D. Both of these paths cross the front face of the cube. Two other paths cross the bottom face, and two more cross the left face. So, there are two (shortest) ways to get to the point B from the point A. Furthermore, instead of crossing the front face of the cube, we may cross the bottom or left sides of the cube. These are the only faces that are adjacent to the vertex A. Hence, there are six such shortest paths.

Answer: There are six shortest paths.

(c) By joining the "closest" corners by line segments, the six corners form a regular hexagon. The figure is formed by joining the closest midpoints of each edge which does not have the A and B as "end" points. The edges containing the points x and y do not have A and B as end points. Since a cube is symmetric, the lengths of each side are the same, and similarly, the angles formed by joining the closest corners are the same. Since the polygon has six vertices, the figure is a (regular) hexagon.

Answer: The figure formed is a (regular) hexagon.