BRITISH COLUMBIA SECONDARY SCHOOL MATHEMATICS CONTEST, 2007

Solutions

Junior Preliminary

1. Since there are 60 students in Mr. Smith's class, and three-quarters of them are girls, there must be 45 girls and 15 boys. Similarly, there must be 15 girls and 30 boys in Ms. Perry's class. This gives a total of 45 boys and 60 girls in the combined class for a boys to girls ratio of 3 : 4.

Answer is (A).

2. Using the familiar formula for the area of a circle, the fraction that is shaded is

$$\frac{\frac{\pi}{2}x^2 - \frac{\pi}{2}\left(\frac{x}{2}\right)^2}{\frac{\pi}{2}\left(\frac{3x}{2}\right)^2} = \frac{1}{3}$$

Answer is (C).

3. By elementary reasoning on isosceles triangles we deduce that the two diagonals intersect at right angles. It follows from Pythagoras' theorem that the perimeter of the rhombus is $4 \times \sqrt{1^2 + 3^2} = 4\sqrt{10}$.

Answer is (C).

4. We need to compute the least common multiple of the given numbers. To do this we calculate the least common multiple using the product of the primes to the maximal powers over all primes in the prime factorizations of the numbers. We have, $2 = 2$, $4 = 2^2$, $6 = 2^1 \times 3^1$, $10 = 2^1 \times 5^1$, $12 = 2^2 \times 3^1$, and $14 = 2^1 \times 7^1$, so the least common multiple is $2^2 \times 3^1 \times 5^1 \times 7^1 = 420$, and the sum of its digits is 6.

Answer is (B).

5. Let x denote the number of passes Alan throws in the remainder of the season. Then $\dfrac{x+6}{x+24} = \frac{4}{5}$, from which we find $x = 66$. So the total number of passes Alan throws over the season is $66 + 24 = 90$.

Alternate solution: If Alan completes all of the rest of his passes, the total number of passes he did not complete is 75% of 24, which is 18. If Alan completes 80% of his passes over the whole season, he misses only 20%, or 1/5, of them. But this is 18 passes, so the total number of passes Alan throws is $5 \times 18 = 90$.

Answer is (E).

6. We compute, $41 + \downarrow 35 \downarrow - \uparrow 53 \uparrow + \Big\uparrow \downarrow 35 \downarrow \Big\uparrow = 41 + 31 - 59 + 37 = 50$.

Answer is (B).

7. Recall, an integer is divisible by 9 if and only if the sum of its digits is divisible by 9. Since the sum of the digits of the integer formed in the way described in the problem is $1 + 2 + 3 + ... + 9 = 45$ which is divisible by 9, it follows that the probability that the integer is divisible by 9 is 1.

Answer is (E).

8. If n is a two digit number then $n = 10a + b$ where $1 \leq a \leq 9$ and $0 \leq b \leq 9$. Then the sum of the digits is $a + b$, and $1 \leq a + b \leq 18$, so division of n by $a + b$ leaves a remainder of at most 17. Working backwards from possible remainders of 17 we eliminate 99, 98, 89. Then we find that 16 into 79 leaves 4 with a remainder of 15 which must be the largest possible.

Answer is (C).

9. Evidently, removing the pieces does not change the perimeter so the side length must be 10 cm, and the area 100 cm^2. The remaining piece must have area 80 cm^2.

Answer is (C).

10. Let C be the centre of the larger circle and let B be the base of the (necessarily) perpendicular line from C to the lower of the two tangents to the two circles. The line segment AC bisects the angle at A, so that triangle ABC is a 30°-60°-90° triangle with the 30° angle at A. Since BC is a radius of the larger circle and it is the side opposite the 30° angle, the hypotenuse AC is 12. If the smaller circle has centre E and radius r, the same argument shows that $\overline{AE} = 2r$. Then $\overline{AC} = 2r + r + 6 = 3r + 6 = 12$ so $r = 2$.

Answer is (A).

11. If we let A denote the area of rectangle $ABCD$, then $A = \left(\frac{33}{5}\right)\left(\frac{14}{3}\right) = \frac{154}{5}$ so the area of triangle AFX is $\left(\frac{5}{8}\right)\left(\frac{154}{5}\right) = \frac{77}{4}$. Then, since triangles AFX and FBX have equal base and height, the area of triangle ABX is twice the area of triangle AFX and so is $2\left(\frac{77}{4}\right) = \frac{77}{2}$. But, the area of triangle ABX is $\left(\frac{1}{2}\right)\left(\frac{33}{5}\right)\left(\overline{BX}\right) = \frac{77}{2}$ so that $\overline{BX} = \left(\frac{77}{2}\right)\left(\frac{10}{33}\right) = \frac{35}{3}$.

Answer is (A).

12. Let X denote the total number of persons in the corner rooms, and let Y denote the total number of persons in the interior side rooms. Then $X + Y = 28$ and, since each corner room is counted twice when counting the total number of people on the sides of the lodge, $2X + Y = 4 \times 9 = 36$, so $X = 8$ and $Y = 20$.

Answer is (D).

Senior Preliminary

1. The number of possible handshakes, counting shakes with neighbors is

$$\binom{20}{2} = \frac{20 \times 19}{2 \times 1} = 190$$

This includes the 20 shakes with neighbors, so the total number of handshakes excluding the shakes with neighbors is $190 - 20 = 170$.

Answer is (A).

2. The number cannot have a single digit, so we are looking for the two digit prime numbers whose have a sum of 10. The pairs of digits that add to 10 are 1 and 9, 2 and 8, 3 and 7, 4 and 6, and 5 and 5. The digits cannot be even and 55 is obviously not prime. So the possibilities are 19, 91, 37, and 73. Of these, all but $91 = 7 \times 13$ is prime. Thus, there are 3 prime numbers less than 100 that have digits that sum to 10.

Answer is (D).

3. If x is the length of the two perpendicular sides of the triangle then, since the area of the triangle is 12.5,
$$\frac{1}{2}x^2 = 12.5 \Rightarrow x^2 = 25 \Rightarrow x = 5$$
By Pythagoras' theorem, the hypotenuse of the triangle is $\sqrt{5^2+5^2} = 5\sqrt{2}$. Hence, the perimeter of the triangle is $5+5+5\sqrt{2} = 5\left(2+\sqrt{2}\right)$. Since $\sqrt{2} \approx 1.4$, the perimeter is approximately $5 \times 3.4 = 17$.

Answer is (C).

4. Using the fact that $28! = 28 \times 27!$ and $29! = 29 \times 28 \times 27!$ we have
$$27! + 28! + 29! = 27!\,(1 + 28 + 29 \times 28) = 27! \times 29 \times (1 + 28) = 27! \times 29 \times 29$$
Hence the largest prime factor of $27! + 28! + 29!$ is 29.

Answer is (B).

5. Since $a^0 = 1$ for any real number $a \neq 0$, one possibility for solving the equation is
$$x^2 - 5x + 6 = 0 \Rightarrow (x-2)(x-3) = 0 \Rightarrow x = 2, x = 3$$
Further, $1^b = 1$ for any real number b, so $x = 1$ is also a solution to the equation. Finally, $(-1)^n = 1$ where n is an even integer, and, for $x = -1$, $x^2 - 5x + 6 = 1 + 5 + 6 = 12$ is even. Thus, $x = -1$ is also a solution. Since these are the only possible solutions to the equation, there is a total of 4 solutions.

Answer is (E).

6. Let n be the number of sides of the polygon and I be the common interior angle. Draw a line from two adjacent vertices on the polygon to the centre to form an isosceles triangle with angles $I/2$, $I/2$ and C, the central angle. Then we have $I + C = 180$ and $C = 360/n$. Now, for I to be an integer, C must be an integer. This is the case if n is a divisor of 360. Since $360 = 2^3 \times 3^2 \times 5^1$, there are $4 \times 3 \times 2 = 24$ divisors of 360. However, these divisors include $n = 1$ and $n = 2$, neither of which give a polygon. Thus, there are 22 divisors of 360 that give polygons for which the common interior angle is an integer. This gives 22 regular polygons that have interior angles which are integers when in degrees.

Answer is (D).

7. The graph of equation (1) is the straight line $y = x + 4$. The graph of equation (2) is the straight line $y = x + 4$ with a hole in the graph at $x = 4$ and $y = 8$. The graph of equation (3) consists of the two lines: $y = x + 4$ and the vertical line $x = 4$. Thus, all of the graphs are different.

Answer is (E).

8. Simplifying the expression gives
$$\begin{aligned}\frac{3^{502} - 3^{500} + 16}{3^{500} + 2} &= \frac{3^{500}\left(3^2 - 1\right) + 16}{3^{500} + 2} \\ &= \frac{8\left(3^{500} + 2\right)}{3^{500} + 2} = 8\end{aligned}$$

Answer is (A).

9. The given equation can be written as $z^2 + z = 8$. Squaring both sides gives $z^4 + 2z^3 + z^2 = 64$. Then $z^4 + 2z^3 + z^2 - 6\left(z^2 + z\right) = z^4 + 2z^3 - 5z^2 - 6z$. Hence,

$$z^4 + 2z^3 - 5z^2 - 6z + 5 = 64 - 6 \times 8 + 5 = 21$$

Note that the solutions to the quadratic $z^2 + z - 8 = 0$ are $z = \dfrac{-1 \pm \sqrt{33}}{2}$. If either of these solutions is plugged into the fourth degree expression above you will get the value 21. Try it for fun.

Answer is (B).

10. Since AB is the diameter of the semi-circle, triangle ABC is an isosceles right triangle. Let x be the radius of the semi-circle, then by Pythagoras' theorem $r = \sqrt{2}x$. Consider the circular wedge ABC. The area of the wedge is $\pi r^2/4$. The area of the triangle ABC with base of $2x$ and altitude x, is x^2. Subtracting this area from the area of the wedge, and this result from the area of the first semicircle, gives the area of the crescent as

$$\frac{\pi}{2}\left(\frac{r}{\sqrt{2}}\right)^2 - \left(\frac{\pi r^2 - 2r^2}{4}\right) = \frac{r^2}{2}$$

Answer is (B).

11. Let v_a Antonino's swimming speed and v_r be Ricardo's. Then $v_r = \frac{2}{3}v_a$. Let L be the length of the pool. After Antonino and Ricardo have swum for a certain time t, measured from when Antonino starts, the total distance each has swum is:

$$D_a = v_a t \quad \text{and} \quad D_r = v_r t + \frac{L}{4} = \frac{2}{3}v_a t + \frac{L}{4}$$

Antonino first overtakes Ricardo when

$$D_a = D_r \Rightarrow v_a t = \frac{2}{3}v_a t + \frac{L}{4} \Rightarrow \frac{1}{3}v_a t = \frac{L}{4} \Rightarrow v_a t = \frac{3}{4}L$$

that is, Antonino first overtakes Ricardo after he has swum three quarters of the length of the pool. Antonino again overtakes Ricardo when he has gone one extra lap, that is, two extra lengths of the pool. This happens when

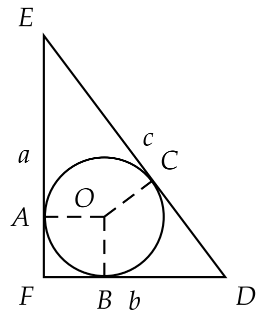
$$D_a = D_r + 2L \Rightarrow v_a t = \frac{2}{3}v_a t + \frac{9L}{4} \Rightarrow \frac{1}{3}v_a t = \frac{9L}{4} \Rightarrow v_a t = \frac{27}{4}L$$

So Antonino next overtakes Ricardo when he as gone six and three quarters lengths, or, six more lengths than the first time he overtook Ricardo. Continuing in the same way we see that Antonino overtakes Ricardo after $\frac{3}{4}$, $6\frac{3}{4}$, $12\frac{3}{4}$, $18\frac{3}{4}$, $24\frac{3}{4}$, $30\frac{3}{4}$, $36\frac{3}{4}$, $48\frac{3}{4}$, and $54\frac{3}{4}$ lengths during his 60 length swim. Thus, Antonino overtakes Ricardo a total of 10 times during his swim.

Answer is (B).

12. Let r be the radius of the circle. Then $\overline{AO} = \overline{BO} = \overline{CO} = \overline{AF} = \overline{BF} = r$. Thus, $\overline{AE} = a - r$ and $\overline{BD} = b - r$. Triangles AEO and CEO are right triangles with two equal sides, the hypotenuse EO and sides $\overline{AO} = \overline{CO} = r$. Hence, they are congruent with $\overline{AE} = \overline{CE} = a - r$. The same argument shows that $\overline{CD} = \overline{BD} = b - r$. Then we have $\overline{CE} = c - \overline{CD} = c - (b - r) = c - b + r$. Equating the two expression for $\overline{CE}$ gives

$$a - r = c - b + r \Rightarrow 2r = a + b - c \Rightarrow r = \tfrac{1}{2}(a + b - c)$$



This completes the solution, but note that the diagram above can be used to prove Pythagoras' theorem. Consider the areas of triangles EFO, DFO, and DEO. These areas are $\frac{1}{2}ar$, $\frac{1}{2}br$, and $\frac{1}{2}cr$, respectively. Their sum is the area of the triangle DEF. Hence, we have

$$\tfrac{1}{2}ab = \tfrac{1}{2}ar + \tfrac{1}{2}br + \tfrac{1}{2}cr \Rightarrow r = \frac{ab}{a+b+c}$$

Equating this expression for r with the one found in the solution gives

$$\tfrac{1}{2}(a + b - c) = \frac{ab}{a+b+c}$$

Cross multiplying gives

$$(a + b)^2 - c^2 = 2ab \Rightarrow a^2 + b^2 = c^2$$

which is Pythagoras' Theorem.

Answer is (A).

Junior Final, Part A

1. Dividing 2007 by 223 shows that $2007 = 223 \times 3^2$. So there are $2 \times 3 = 6$ factors.

Answer is (D).

2. Since $AB = 2$, by successive use of Pythagoras' theorem $AC = 2\sqrt{2}$, $AD = 2\sqrt{3}$, $AE = 2\sqrt{4}$, and $AF = 2\sqrt{5}$.

Answer is (D).

3. First note that $2L \times 1.5W = 3LW = 60 \Rightarrow LW = 20$. We get maximum perimeter if the rectangle is as elongated as possible, $L = 20$, $W = 1$ which makes the resulting rectangle $40\,\text{cm} \times 1.5\,\text{cm}$. The maximum perimeter is $83\,\text{cm}$.

Answer is (E).

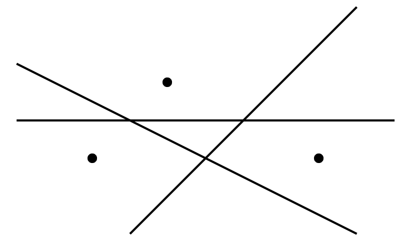
4. The triangle lies in a rectangle with vertices $(1, 2)$, $(5, 2)$, $(5, 7)$, and $(1, 7)$ which has area 20. But then we have to subtract the area of the three triangles with vertices: $(1, 2)$, $(5, 2)$, and $(5, 4)$ with area 4; $(5, 4)$, $(5, 7)$ and $(4, 7)$ with area $\dfrac{3}{2}$; and $(4, 7)$, $(1, 7)$, and $(1, 2)$ with area $\dfrac{15}{2}$. So the area of the given triangle is $20 - 4 - \dfrac{3}{2} - \dfrac{15}{2} = 7$.

Answer is (A).

5. First $8 \star 6 = 24$. Then $20 \star (8 \star 6) = 20 \star 24 = 120$.

Answer is (B).

6. For a line to be equidistant from two points it must be either parallel to the line through the two points or through the midpoint between them. For three points there are three possibilities:



Answer is (C).

7. First, concentrate on the units position; you will have 0009, 0019, ..., 4999 for a total of 500 times. Now look at the tens position; we have 009*, 019*, ..., 499* where $* = 0, 1, \ldots, 9$ (ten times) for a total of $50 \times 10 = 500$ times. Next for the hundreds position; we have 09**, 19**, ..., 49** where $** = 00, 01, \ldots, 99$ (one hundred times) for a total of $5 \times 100 = 500$. Thus we write 9 a total of $500 + 500 + 500 = 1500$ times.

Answer is (D).

8. Draw the line segment EO. Then $\triangle DEO$ and $\triangle AEO$ are isosceles triangles, so that $\angle EDO = \angle EOD$ and $\angle OEA = \angle OAE$. Further, $\angle OEA = 2\angle EDO$, and $\angle OEA + \angle OAE = \angle AOB + \angle EOD \Rightarrow \angle AOB = 3\angle EDO$. Hence, if $\angle AOB = 45^\circ$, then $\angle EDO = 15^\circ$ so $\angle OAE = 2\angle EDO = 30^\circ$.

Answer is (C).

9. If we put the six chairs for the students in a row, then there are five spaces between them for the professors to choose. This can be done in $5 \times 4 \times 3 = 60$ ways.

Answer is (D).

10. Jim must answer "yes" if either he is a Zorian or both he and Steve are Lannites. He will answer "no" if he is a Lannite and Jim is a Zorian. Jim must have answered "yes" or Sam would have known the two religions. Now Steve will answer "yes" if they both have the same religion and "no" otherwise. There is still an ambiguity if they have the same religion, so Steve must have answered "no" and Jim is a Zorian and Steve is a Lannite.

Answer is (C).

Junior Final, Part B

1. Let N be the number of nickels, D be the number of dimes and Q be the number of quarters. Since the total value of Joan's collection is \$2.00, we have $5N + 10D + 25Q = 200$, or $N + 2D + 5Q = 40$. Now, if nickels were dimes and dimes were nickels, then the value of the collection is \$1.70. In other words $10N + 5D + 25Q = 170$, or $2N + D + 5Q = 34$. These equations can be solved to yield

$$N = \frac{28 - 5Q}{3}, \tag{1}$$

$$D = \frac{46 - 5Q}{3}. \tag{2}$$

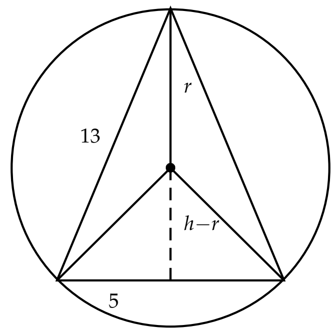
In view of equation (1) the number of quarters cannot exceed five. As well, since there must be an integer number of coins, the only possibilities for the quarters are $Q = 2, 5$. It follows that $(N, D, Q) = (6, 12, 2), (1, 7, 5)$.

Answer: There can be either 6 nickels, 12 dimes and 2 quarters, or 1 nickel, 7 dimes, and 5 quarters.

2. The number of $1 \times 1 \times 1$ cubes is 27, the number of $2 \times 2 \times 2$ cubes is 8, and there is a single $3 \times 3 \times 3$ cube. The total number of cubes is $27 + 8 + 1 = 36$.

Answer: The total number of cubes with sides of integral length is 36.

3. Let h be the height of the triangle, then $h^2 + 5^2 = 13^2$ or $h^2 = 144$. So that $h = 12$. Now, from the diagram we have $(h - r)^2 + 25 = r^2$. Since $h = 12$, this can be simplified to yield $144 - 24r + r^2 + 25 = r^2$, which gives $r = 169/24$.



Answer: The radius of the circumscribed circle is $\frac{169}{24}$.

4. Label the rows and columns as shown. First observe that for any of the letters given along the edges, that letter must appear in either the first or second square from the edge. Then the C in column d must be at 4d, otherwise the B will not be the last letter in row 3. This means that the C in column b must be at 3b. There must be a C in column a and it cannot be in rows 3 or 4, since there is already a C in these rows, or in row 2 to accommodate the A. Hence, there must be a C at 1a and 2c. The B in column a must be at 4a, since the other squares in this column already contain a letter or cannot have a B. A similar argument shows that the A in column d must be at 1d. The A in row 4 must be at 4c, since 4b must be the blank in the row and the other two squares are occupied. Now the A in row 3 must be at 3a, since there is already an A in column c and there cannot be an A at 3d to accommodate the B. The last A goes at 2b. Finally the B's go at 1b, 3c, and 2d. The final, unique, solution is shown above.

		a	b	c	d	
1		C	B		A	
2	A		A	C	B	
3		A	C	B		B
4		B		A	C	
			C		C	

Answer: Shown in grid above.

5. First note that $x < 12$. So x can only be an integer between 1 and 11. If L is the least common multiple of x and 12, then there are integers n and m for which $L = xn = 12m$. So that solving for y gives

$$\frac{1}{y} = \frac{1}{x} - \frac{1}{12} = \frac{n-m}{L}$$

If $n - m$ is a divisor of L, then y is an integer. For $x = 11$, $L = 11 \times 12$, $m = 11$, and $n = 12$, so that $n - m = 1$ and $y = 11 \times 12 = 132$. For $x = 10$, $L = 5 \times 12$, $m = 5$, and $n = 6$, so that $n - m = 1$ and $y = 5 \times 12 = 60$. Continuing in the same way, as shown in the table

x	11	10	9	8	6	4	3
L	132	60	36	24	12	12	12
m	11	5	3	2	1	1	1
n	12	6	4	3	2	3	4
y	132	60	36	24	12	6	4

Thus, there is a total of 7 pairs of positive integers, (x, y). From the table above they are: $(11, 132)$, $(10, 60)$, $(9, 36)$, $(8, 24)$, $(6, 12)$, $(4, 6)$, and $(3, 4)$.

Alternate solution: Write the equation above as

$$12(y - x) = xy \Rightarrow 12y - 12x - xy + 12^2 = 12^2 \Rightarrow (12 - x)(12 + y) = 12^2$$

Thus, the factors $(12 - x)$ and $(12 + y)$ must be a divisors of $12^2 = 144$ that multiply to 144. Further, for both x and y to be positive, we must have $x < 12$. So the only possible values for $12 - x$ are: 1, 2, 3, 4, 6, 8, and 9. Solving for x gives the values in the top row of the table above. The corresponding values of $12 + y$ are: 144, 72, 48, 36, 24, 18, and 16. Solving for y gives the values in the last row of the table above.

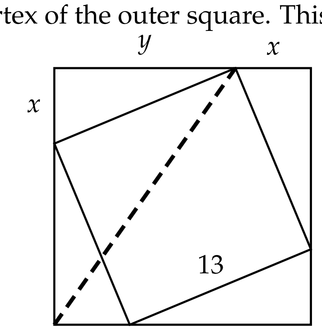
Answer: The pairs (x, y) of positive integer solutions are: $(11, 132)$, $(10, 60)$, $(9, 36)$, $(8, 24)$, $(6, 12)$, $(4, 6)$, $(3, 4)$.

Senior Final, Part A

1. Since $10000 = 100^2$, there are 99 squares less than 10000.

Answer is (A).

2. With the diagram as labeled, we have $x + y = 17$ and $x^2 + y^2 = 13^2 = 169$. Squaring the first of these equations and subtracting the second gives $2xy = 120 \Rightarrow xy = 60$. Solving for y and substituting into the first equation gives the quadratic equation $x^2 - 17x + 60 = (x - 5)(x - 12) = 0$. This gives $x = 5$ and $y = 12$. The length of the heavier dashed line in the diagram is the greatest distance between a vertex of the inner square and a vertex of the outer square. This distance is $\sqrt{12^2 + 17^2} = \sqrt{433}$.



Answer is (C).

3. We want the edge and the corner cubes. There are 12 edges, each with $n-2$ non-corner cubes for a total of $12 \times (n-2)$ non-corner edge cubes. Then there are 8 corner cubes, so the total we are looking for is $12 \times (n-2) + 8 = 12n - 16 = 4(3n-4)$.

   Alternate solution: The total number of edge cubes is $12n$. Since each corner cube is on three edges, this counts each corner cube three times. So we subtract each corner cube twice to give $12n - 2 \times 8 = 4(3n-4)$.

   Answer is (B).

4. If a, b, and c are roots of the equation $x^3 + kx^2 - 1329x - 2007 = 0$, then

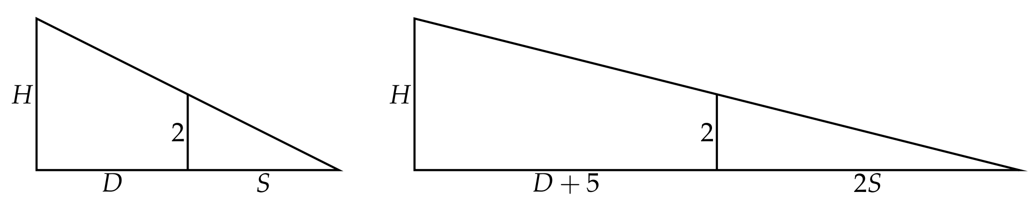
$$(x-a)(x-b)(x-c) = x^3 - (a+b+c)x^2 + (ab+ac+bc)x - abc = x^3 + kx^2 - 1329x - 2007$$

   So we must have $abc = 2007 = 3^2 \times 223$. The possibilities are $a = \pm 1$, $b = \pm 9$, $c = \pm 223$, $a = \pm 1$, $b = \pm 3$, $c = \pm 669$, $a = \pm 1$, $b = \pm 1$, $c = \pm 2007$, or $a = \pm 3$, $b = \pm 3$, $c = \pm 223$. Checking these possibilities shows that the only case that gives $ab + ac + bc = -1329$ and $abc = 2007$ is $a = -3$, $b = -3$, and $c = 223$. This gives $k = -(a+b+c) = -217$.

   Answer is (E).

5. Let D be the distance from the lamppost to the man (the quantity which we want), S the length of his shadow, and H the height of the lamppost. Then by similar triangles we have

$$\begin{aligned} \frac{2}{S} &= \frac{H}{D+S} \\ \frac{2}{2S} &= \frac{H}{D+5+2S} \\ HS = 2(D+S) &= D+5+2S \\ D &= 5 \end{aligned}$$

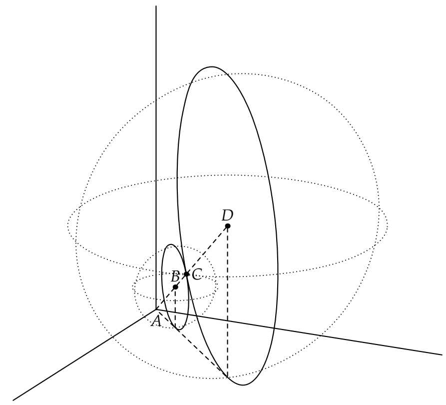


   Answer is (A).

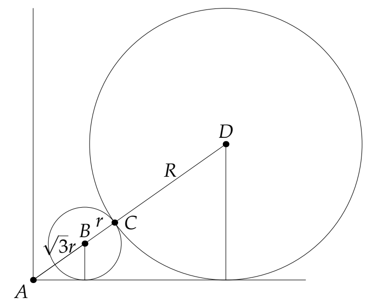
6. First, concentrate on the units position; you will have 0009, 0019, ..., 5549 for a total of 555 times. Now look at the tens position; we have 009*, 019*, ..., 549* where $* = 0, 1, \ldots, 9$ (ten times) for a total of $55 \times 10 = 550$ times. Next for the hundreds position; we have 09**, 19**, ..., 49** where $** = 00, 01, \ldots, 99$ (one hundred times) for a total of $5 \times 100 = 500$. Thus we write 9 a total of $555 + 550 + 500 = 1605$ times.

   Answer is (E).

7. Let R be the radius of the basketball and r the radius of the softball. In the diagrams on the right A is the corner where the three walls meet. If this is the origin of a three dimensional coordinate system with axes along the three lines of intersection of the walls, the centre of the basketball D has coordinates (R, R, R), and the centre of the softball B has coordinates (r, r, r). Using the three dimensional form of Pythagoras' theorem, the distance from A to D is $\sqrt{3}R$ and the distance from A to B is $\sqrt{3}r$. From the bottom diagram on the right we see that the length of AD can be broken into three parts: $\overline{AB} = \sqrt{3}r$, $\overline{BC} = r$, and $\overline{CD} = R$. Thus



$$R + r + \sqrt{3}r = \sqrt{3}R$$



and solving for r gives

$$r = R\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right) = R\left(2-\sqrt{3}\right)$$

Taking $R = 20$ gives

$$r = 20\left(2-\sqrt{3}\right)$$

Answer is (A).

8. The probability that Eve gets her channel the first try is $\frac{1}{10}$, for the second try it is $\frac{9}{10} \times \frac{1}{10} = \frac{9}{100}$ and for the third try it is $\frac{9}{10} \times \frac{9}{10} \times \frac{1}{10} = \frac{81}{1000}$. Thus the probability that she will take no more than three tries is $\frac{1}{10} + \frac{9}{100} + \frac{81}{1000} = \frac{271}{1000}$.

Answer is (B).

9. Let $r = x/y$ and $s = z/y$. Then dividing numerator and denominator of the first two expressions by y gives $1/(r-s) = (r+1)/s = r$. Multiplying the first and third equations by $r-s$ gives $1 = r^2 - rs$ and multiplying the second and third equations by s gives $r + 1 = rs$. Substituting for rs into the first equation gives $1 = r^2 - r - 1 \Rightarrow r^2 - r - 2 = (r-2)(r+1) = 0$. The solutions to this equation are $r = 2$ or $r = -1$. If $r = -1$, then $rs = 0 \Rightarrow s = 0$. But x, y, and z are all positive, so this is impossible. Hence, $r = x/y = 2$.

Alternate solution: Note that

$$\frac{a}{b} = \frac{c}{d} \Rightarrow \frac{c}{a} = \frac{d}{b} = x \Rightarrow c = ax \;\&\; d = bx$$

So that

$$\frac{a+c}{b+d} = \frac{a+ax}{b+bx} = \frac{a(1+x)}{b(1+x)} = \frac{a}{b}$$

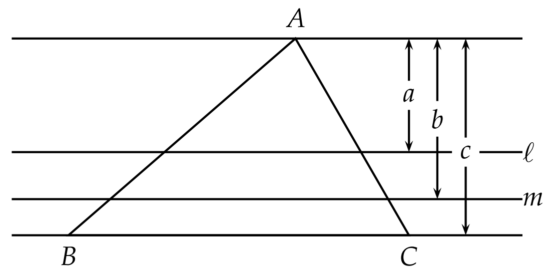
Thus, if two or more fractions have a common value, then dividing the sum of the numerators by the sum of the denominators gives a fraction with same value. Applying this result to the fractions above gives

$$\frac{x}{y} = \frac{y+x+y+x}{x-z+z+y} = \frac{2x+2y}{x+y} = 2$$

as derived above.

Answer is (D).

10. We have three similar triangles whose areas are proportional to their altitudes. The areas are in proportion 1 : 2 : 3 so the altitudes must be in proportion



$$a : b : c = 1 : \sqrt{2} : \sqrt{3} = 10\frac{\sqrt{3}}{3} : 10\frac{\sqrt{6}}{3} : 10$$

Thus the distance between lines ℓ and m is

$$10\frac{\sqrt{6}}{3} - 10\frac{\sqrt{3}}{3} = 10\frac{\sqrt{3}}{3}\left(\sqrt{2}-1\right)$$

Answer is (D).

Senior Final, Part B

1. Let x be the other perpendicular leg and let y be the hypotenuse, then $x^2 + 11^2 = y^2$. This equation can be rewritten as $y^2 - x^2 = 121$ or $(y-x)(y+x) = 121$. Since all the sides must be of integer length, last equation implies that both $(y-x)$ and $(y+x)$ must be integer factors of 121. Let p and q be any two integer factors of 121. That is $121 = pq$. We have the system of equation

$$\begin{aligned} y - x &= p\,, \\ y + x &= q\,, \end{aligned}$$

which has the solution

$$\begin{aligned} y &= = \frac{p+q}{2}\,, \\ x &= = \frac{q-p}{2}\,. \end{aligned}$$

The factors of 121 are 1, 11 and 121. The possibilities for p and q are $(p,q) = (1,121), (11,11), (121,1)$. The first implies $x = -60$, while the second implies $x = 0$. The third possibility is the only one that leads to positive integer solutions for x and y. We conclude that $(x,y) = (60,61)$.

Answer: The lengths of the other two sides of the triangle are 60 and 61.

2. Consider the first three pairs of sums

$$\begin{gathered} 1+2=3 \\ 4+5+6=7+8 \\ 9+10+11+12=13+14+15 \\ \vdots \end{gathered}$$

(a) When we consider the first three sums, the number of addends on the LHS is 2, 3, 4 and the number of addends on the RHS is 1, 2, 3. This implies that the tenth sum will have 11 addends on the LHS and 10 addends on the RHS. In other words, the tenth sum will be of the form

$$n + (n+1) + \cdots + (n+10) = (n+11) + (n+12) + \cdots + (n+20)\,,$$

for some integer positive integer n. The largest number on the RHS is $(n+20)$. To find this number, we solve the preceding equation for n. This equation can be written as $11n + 1 + 2 + \cdots + 10 = 10n + 11 + 12 + \cdots + 20$ or, if we solve for n, we obtain

$$\begin{aligned} n &= \sum_{j=11}^{20} j - \sum_{j=1}^{10} j = \sum_{j=1}^{20} j - 2\sum_{j=1}^{10} j \\ &= \frac{20 \times 21}{2} - 10 \times 11 = 100 \end{aligned}$$

It follows that the largest number of the RHS of the tenth sum is $100 + 20 = 120$.

Alternate solution: The n^{th} row begins with n^2, so row 11 begins with $11^2 = 121$. This implies that row 10 ends with 120.

Answer: The largest number on the right hand side of the tenth pair of sums is 120.

(b) The sum on the LHS is $11n + \sum_{j=1}^{10} j = 1100 + 5 \times 11 = 1155$, while the sum on the RHS is $10n + \sum_{j=11}^{20} j = 10n + \sum_{j=1}^{20} j = -\sum_{j=1}^{10} j = 1000 + 10 \times 21 - 5 \times 11 = 1155$ as expected.

Alternate solution: The sum of the entries in on row n is the sum of the integers from n^2 to $(n+1)^2 - 1$. The sum of the entries on either side of the equation is half of this. For row 10 this is

$$\frac{1}{2}(100 + 101 + \cdots + 120) = \frac{1}{2}\left(2100 + \frac{(20)(21)}{2}\right) = 1155$$

Answer: The value of the sum on each side of the tenth pair of sums is 1155.

(c) The k^{th} sum will be of the form

$$n_k + (n_k + 1) + \cdots + (n_k + k) = (n_k + [k+1]) + (n_k + [k+2]) + \cdots + (n_k + 2k)\,,$$

or

$$(k+1)\,n_k + \sum_{j=1}^{k} j = k\,n_k + \sum_{j=1}^{2k} j - \sum_{j=1}^{k} j\,.$$

If we solve the above for n_k, then we find that

$$n_k = \sum_{j=1}^{2k} j - 2\sum_{j=1}^{k} j = \frac{2k(2k+1)}{2} - k(k+1) = k^2\,.$$

A formula for the sum on the RHS is

$$S_{\text{RHS}} = (k+1)k^2 + \frac{k(k+1)}{2} = \frac{k(k+1)(2k+1)}{2}\,,$$

while a formula for the sum on the LHS is

$$S_{\text{RHS}} = k^3 + \frac{2k(2k+1)}{2} - 2\frac{k(k+1)}{2} = \frac{k(k+1)(2k+1)}{2}\,,$$

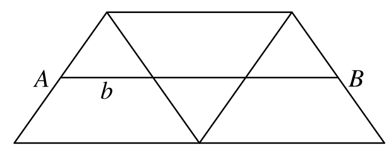
as expected.

Alternate solution: Using the observation in part (b) we have

$$\begin{aligned}
\frac{1}{2}\left[k^2 + \left(k^2+1\right) + \cdots + \left((k+1)^2 - 1\right)\right] &= \frac{1}{2}\left(k^2 + k^2 + 1 + \cdots + k^2 + 2k\right) \\
&= \frac{1}{2}\left[k^2(2k+1) + \frac{2k(2k+1)}{2}\right] \\
&= \frac{k(2k+1)(k+1)}{2}
\end{aligned}$$

Answer: The expression $\frac{k(k+1)(2k+1)}{2}$ gives the sum on each side of the k^{th} pair of sums.

3. Suppose we "flatten" the pyramid. If we refer to the diagram on the right, which shows three of the four faces of the flattened pyramid, then we see that that the shortest distance from point A to point B is a straight line through the midpoints of the three triangles. Now by similar triangles, the base b of the smaller triangles, formed by the ant's path, satisfies $b/10 = 5/10$. Hence $b = 5$. Therefore, the shortest distance from point A to point B is 15.



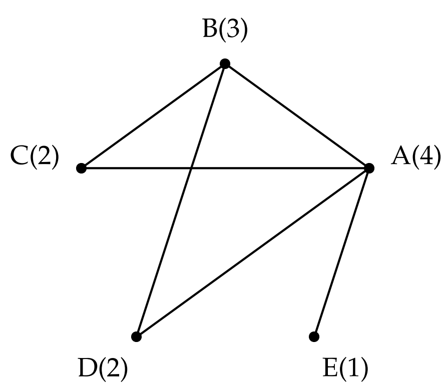
Answer: The length of the shortest path the ant has to walk is 15 cm.

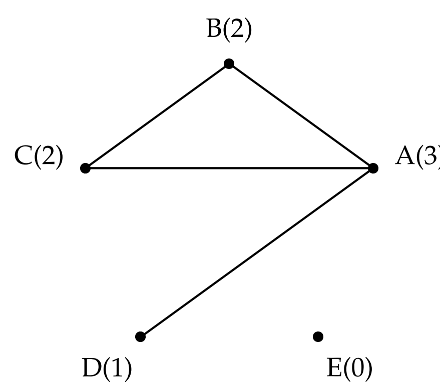
4. Five people are trapped on the island of Survival. Each person can form an alliance with any of the other people on the island. The alliances are mutual, so that if A is allied with B, then B is allied with A. However, if A is allied with B and B is allied with C, it is not necessarily true that A and C are allied. If every person on the island is allied with every other person on the island, then there are ten pairs of people on the island who each have the four alliances.

   (a) A person can have no more than 4 alliances, but could have zero alliances. Thus there are only five possible values for the number of alliances a person can have: 0, 1, 2, 3, 4. Is it possible that no two people have the same number of alliances? If it were, then one person has 0 alliances, one has 1, one has 2, one has 3, and one has 4. However, if one person has 4 alliances, then that person has an alliance with every other person, so no person can have 0 alliances. Hence, it is not possible that no two people have the same number of alliances.

   Answer: It is not possible for no two people to have the same number of alliances.

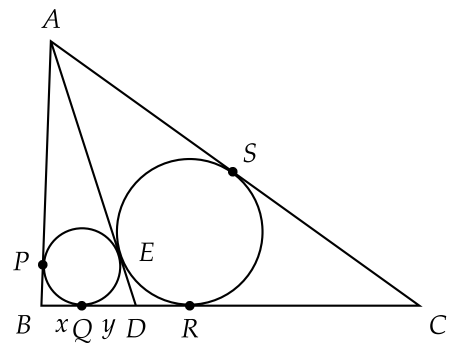
   (b) It is possible to have exactly one pair with the same number of alliances. Suppose that A has an alliance with everyone else and that B has an alliance with C and D in addition to the alliance with A. Now A has 4 alliances, B has 3, both C and D have two alliances, and E has one alliance. Thus, the smallest number of pairs that can have the same number of alliances is 1. There is at least on other case that gives exactly one pair with the same number of alliances. Suppose that A has an alliance with all others except E and B has an alliance with C in addition to the alliance with A. Then A has 3 alliances, B and C have 2 alliances, D has 1 alliance, and E has 0 alliances. These two cases can be represented by the graphs shown below. See if you can find another case, or prove that there is no other case, that gives exactly one pair that has the same number of alliances.





Answer: It is possible that only one pair of people have the same number of alliances.

5. Starting with the point E and moving counterclockwise, we label the points of tangency for the circle on the left as P and Q. In a similar fashion, we label the points of tangency for the circle on the right as R and S. Now, let the $\overline{BQ} = x$ and, since vertex B is common, $\overline{BP} = x$. Now $\overline{AP} = 7 - x$. Since vertex A is common, $\overline{AE} = \overline{AS} = 7 - x$. It follows that $\overline{CS} = 12 - (7 - x) = 5 + x$ hence $\overline{CS} = 5 + x$.
If we let $\overline{DQ} = y$, then $\overline{DE} = \overline{DR} = y$. It follows that $x + y + y + 5 + x = 10$ or $\overline{BD} = x + y = \dfrac{5}{2}$.



Answer: The length of line segment BD is $\frac{5}{2}$.